Anomalous Statistics, Generalized Entropies, and Information Geometry
March 6-10, 2012
Memorial Hall, Nara Women's University, Nara, Japan

Studies on deformed exponential families

1. Extended ϕ-exponential families
2. Nonparametric ϕ-exponential families
3. ϕ-exponential martingales

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Session 13 March 9 16:10–16:50

Introduction

Acknowledgments

- J. Naudts, *Generalised Thermostatistics* (Springer, 2011)

Deformed exponentials I

H1 The real function $\phi \colon \mathbb{R}_> \to \mathbb{R}_>$ is strictly positive, strictly increasing, continuous.

ϕ-logarithm

$$\ln_\phi(v) = \int_1^v \frac{dx}{\phi(x)}, \quad v \in \mathbb{R}_>.$$

Properties

$$\ln_\phi \colon \mathbb{R}_> \to \left] -\int_0^1 \frac{dx}{\phi(x)}, \int_1^{+\infty} \frac{dx}{\phi(x)} \right[=]-m, +M[$$

is strictly increasing, strictly concave and twice differentiable.

ϕ-exponential is the inverse function

$$\exp_\phi = \ln_\phi{}^{-1} \colon]-m, +M[\to \mathbb{R}_>.$$

It is positive, increasing, convex, twice differentiable.

Deformed exponentials II

Rate function The ϕ-exponential is the solution of the Chauchy problem

$$\begin{cases} y'(u) = \phi(y(u)) \\ y(0) = 1 \end{cases},$$

It is convenient to introduce the *rate function*

$$\gamma(u) = \frac{d}{du} \log\left(\exp_\phi(u)\right) = \frac{\phi(\exp_\phi(u))}{\exp_\phi(u)},$$

Derivatives

$$\begin{aligned} {\exp_\phi}'(u) &= \phi(\exp_\phi(u)) = \gamma(u) \exp_\phi(u) . \\ {\exp_\phi}''(u) &= \gamma'(u) \exp_\phi(u) + \gamma(u) {\exp_\phi}'(u) \\ &= (\gamma'(u) + \gamma^2(u)) \exp_\phi(u) . \\ \frac{(\gamma'(u) + \gamma^2(u))}{\gamma(u)} &= \frac{{\exp_\phi}''(u)}{{\exp_\phi}'(u)} = \frac{d}{du} \log\left({\exp_\phi}'(u)\right) . \end{aligned}$$

Deformed exponentials III

Self duality: The deformed exponential is self-dual,

$$\exp_\phi(u)\exp_\phi(-u) = 1, \quad \ln_\phi(v) + \ln_\phi\left(\frac{1}{v}\right) = 0,$$

if, and only if, the rate function γ is symmetric.

H2: We assume ϕ defined on $\mathbb{R}_+$ and

$$\phi(0) = 0, \quad M = \int_1^{+\infty} \frac{dx}{\phi(x)} = +\infty.$$

Extension: The extended $\exp_\phi$ is nonnegative, nondecreasing, convex, differentiable, with derivative $\exp_\phi'(u) = \phi(\exp_\phi(u))$. The rate function in not defined on $]-\infty, -m]$ because there $\exp_\phi(u) = 0$. However,

$$\exp_\phi'(u) = \gamma(u)\exp_\phi(u)$$

if γ is extended with arbitrary bounded values, for example 0, on $]-\infty, -m]$, $\exp_\phi'(u)(\exp_\phi(u))^+ = \gamma(u)$.

Part 1
Extended ϕ-exponential families

- G. Pistone, The European Physical Journal B Condensed Matter Physics **71**(1), 29 (2009), ISSN 1434-6028, `http://dx.medra.org/10.1140/epjb/e2009-00154-y`
- L. Malagò, G. Pistone (2010), arXiv:1012.0637v1
- G. Pistone (2011), arXiv:1112.5123v1

Marginal polytope I

- On the finite state space $(\mathcal{X}, \mu)$ we consider the ϕ-exponential family

$$p_\theta(x) = \exp_\phi\left(\sum_{j=1}^m \theta_j H_j(x) - \alpha(\theta)\right) p(x), \quad \theta \in \mathbb{R}^m.$$

- The function $\alpha\colon \mathbb{R}^m \to \mathbb{R}$ is convex and

$$\alpha(\theta) = \sum_{j=1}^m \theta_j \mathrm{E}_p\left[H_j\right] - \mathrm{E}_p\left[\ln_\phi\left(\frac{p_\theta}{p}\right)\right].$$

- If the convex conjugate

$$\alpha^*(\eta) = \sup\left\{\theta \cdot \eta - \alpha(\theta)\colon \theta \in \mathbb{R}^m\right\}$$

is a maximum value, then

$$\alpha^*(\eta) = \hat{\theta} \cdot \eta - \alpha(\hat{\theta}), \quad \eta = \nabla\alpha(\hat{\theta}).$$

Marginal polytope II

Definition

The *marginal polytope* of the ϕ-model (also called *convex support*) is the convex hull M of the set $\{H(x) \colon x \in \mathcal{X}\} \subset R^m$, $H = (H_1, \ldots, H_m)$.

Example (No-3-way-interaction)

- $\mathcal{X} = \{+1, -1\}$, $\mu = \#$, $p(x) = 1$.
- $$\ln_\phi(p_\theta(x)) = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_{12} x_1 x_2 + \theta_{13} x_1 x_3 + \theta_{23} x_2 x_3 - \alpha(\theta)$$
- The marginal polytope is the convex subset of $\mathbb{R}^6$ with vertices
  $$\{(x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3) \colon x_1, x_2, x_3 = \pm 1\}.$$
- Facets of the marginal polytope can be computed.

Convex conjugate

Theorem

1. *The convex conjugate* $\alpha^* \colon \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ *of* α *is finite exactly on the marginal polytope* $M = \operatorname{co}(\operatorname{im} H)$*.*
2. *The gradient mapping* $\nabla\alpha \colon \mathbb{R}^m \to \mathbb{R}^m$ *is onto the interior* M° *of the marginal polytope* M*.*
3. α^* *restricted to* M° *is the Legendre transform of* α *that is,* $\alpha^*(\eta) = \theta \cdot \eta - \alpha(\theta)$ *if* $\eta = \nabla\alpha(\hat{\theta})$*.*

Proof.

See the ArXiv paper with the assumption $m = +\infty$. Cfr. Th. 3.6 of L.D. Brown, *Fundamentals of statistical exponential families with applications in statistical decision theory*, Number 9 in IMS Lecture Notes. Monograph Series (Institute of Mathematical Statistics, Hayward, CA, 1986), ISBN 0-940600-10-2. □

Non parametric version I

Identification Two different sets of statistics H_j, $j = 1, \ldots, m$ and H'_j, $j = 1, \ldots, m'$ define the same statistical model if, and only if, the vector space generated by the centered random variables is the same,

$$\mathrm{Span}\left(H_j - \mathrm{E}_p\left[H_j\right], j = 1, \ldots, m\right) = \\ \mathrm{Span}\left(H'_j - \mathrm{E}_p\left[H'_j\right], j = 1, \ldots, m'\right).$$

Non parametric Without reference to a vector basis nor to pameters, the ϕ-exponential model is the set of probability density function p_u of the form

$$p_u = \exp_\phi\left(u - K_p(u)\right)p, \quad u \in V,$$

where V is a linear sub-space of $L_0(p)$ and

$$K_p(u) = \alpha(\theta) - \sum_{j=1}^{m} \theta_j \mathrm{E}_p\left[H_j\right], \quad u = \sum_{j=1}^{m} \theta_j \left(H_j - \mathrm{E}_p\left[H_j\right]\right)$$

Non parametric version II

Chart The random variable $u \in V$ is a unique parameterization of p_u as

$$u = \ln_\phi\left(\frac{q}{p}\right) - \mathrm{E}_p\left[\ln_\phi\left(\frac{q}{p}\right)\right].$$

Cumulant The quantity $K_p(u)$ is a divergence of p from p_u, as $K_p(0) = 0$ and, from

$$u - K_p(u) = \ln_\phi\left(\frac{q}{p}\right), \quad u \in L_0(p),$$

we have

$$\begin{aligned} K_p(u) &= -\mathrm{E}_p\left[\ln_\phi\left(\frac{p}{q}\right)\right] \\ &= \mathrm{E}_p\left[\ln_{\tilde{\phi}}\left(\frac{p}{q}\right)\right], \quad \tilde{\phi}(x) = x^2\phi\left(\frac{1}{x}\right) \\ &> \mathrm{E}_p\left[\frac{q}{p} - 1\right] = 0, \quad q \neq p \end{aligned}$$

Non parametric version III

$\mathrm{D}\,K_p\,(u)\,v$ The non parametric derivative of the mapping $L_0(p)\colon u \mapsto K_p(u)$ is the directional derivative in direction $v \in L_0(p)$. With the notation

$$\mathrm{D}\,K_p\,(u)\,v = \left.\frac{d}{dt}K_p(u+tv)\right|_{t=0},$$

one finds

$$\mathrm{E}_p\left[\phi\left(\frac{p_u}{p}\right)v\right] = \mathrm{E}_p\left[\phi\left(\frac{p_u}{p}\right)\right]\mathrm{D}\,K_p\,(u)\,v.$$

Escort The escort mapping

$$\phi_p\colon q \mapsto \frac{\phi\left(\frac{q}{p}\right)}{\mathrm{E}_p\left[\phi\left(\frac{q}{p}\right)\right]}$$

is one-to-one and

$$\mathrm{D}\,K_p\,(u)\,v = \mathrm{E}_p\left[\phi_p(p_u)v\right]$$

Non parametric version IV

$D^2K_p(u)vw$ The second derivative of $u \mapsto \exp_\phi(u - K_p(u))$ in the directions v and w is the first derivative in the direction w of $u \mapsto {\exp_\phi}'(u - K_p(u))(v - DK_p(u)v)$, therefore

$$D^2K_p(u)vw = \frac{\mathrm{E}_p\left[{\exp_\phi}''(u - K_p(u))(v - DK_p(u)v)(w - DK_p(u)w)\right]}{\mathrm{E}_p\left[{\exp_\phi}'(u - K_p(u))\right]}.$$

If $w = v \neq 0$, then $D^2K_p(u)vv > 0$, therefore the functional K is strictly convex.

Conjugation The convex conjugate of $L_0(p) \colon u \mapsto K_p(u)$, is defined in the duality $(u^*, u) \mapsto \mathrm{E}_p[u^* u]$ by

$$H_p(u^*) = \sup\left\{\mathrm{E}_p[u^* u] - K_p(u) \colon u \in L_0(p)\right\}, \quad u^* \in L_0(p).$$

Normal equations

If a maximum is reached at $\hat{u}$, then the directional derivative of

$$L_0(p)\colon u \mapsto \mathrm{E}_p\left[u^* u\right] - K_p(u)$$

is zero in each direction v,

$$\mathrm{E}_p\left[u^* v\right] - \mathrm{E}_p\left[\phi_p(p_{\hat{u}})v\right] = 0, \quad v \in L_0(p).$$

Hence $u^* + 1 = \phi_p\left(p_{\hat{u}}\right)$, therefore

$$\begin{aligned} H_p(u^*) &= \mathrm{E}_p\left[u^* \hat{u}\right] - K_p(\hat{u}) \\ &= \mathrm{E}_p\left[(1+u^*)\hat{u}\right] - \mathrm{E}_p\left[(1+u^*)K_p(\hat{u})\right] \\ &= \mathrm{E}_p\left[(1+u^*)\ln_\phi\left(\frac{p_{\hat{u}}}{p}\right)\right] \end{aligned}$$

Divergence and characterization

Theorem

1. *The convex conjugate* H_p *of* K_p *is finite at* u^* *if, and only if,* $q = (u^* + 1)p$ *is a density, that is* $\mathrm{E}_p[u^*] = 0$, $u^* + 1 \geq 0$.
2. *If* q *is a strictly positive density function and* $u^* = \frac{q}{p} - 1$, *the normal equation is* $1 + u^* = \frac{q}{p} = \phi_p(p_{\hat{u}})$, *hence* $\frac{p_{\hat{u}}}{p} = \phi_p^{-1}(q)$ *and*

$$H_p(q) := H_p\left(\frac{q}{p} - 1\right) = \mathrm{E}_q\left[\ln_\phi\left(\phi_p^{-1}(q)\right)\right].$$

3. *If* $u_1, \ldots, u_d$ *are random variables in* $L_0(p)$ *and* p_θ *is the associates* ϕ*-exponential family, define the escort model* $q_\theta = \phi_p(p_\theta)$. *Let* $\widetilde{q}_\theta$ *be a model wich gives to each* u_j *the same expected value as* q_θ. *Then* $H_p(q_\theta) \leq H_p(\widetilde{q}_\theta)$.

Proof

1. Take a basis and apply the marginal polytope theorem.
2. If q is a strictly positive density function and $u^* = \frac{q}{p} - 1$, the normal equation is $1 + u^* = \frac{q}{p} = \phi_p(p_{\hat{u}})$, hence $\frac{p_{\hat{u}}}{p} = \phi_p^{-1}(q)$ and use the normal equations to get

$$\begin{aligned} H_p(q) &:= H_p(u^*) \\ &= \mathrm{E}_p\left[\frac{q}{p} \ln_\phi\left(\phi_p^{-1}(q)\right)\right] \\ &= \mathrm{E}_q\left[\ln_\phi\left(\phi_p^{-1}(q)\right)\right]. \end{aligned}$$

3. Compare in the definition of conjugate the case with inequality with the case of equality.

! ϕ_p^{-1} is not of the same type as ϕ_p.

Exposed subsets

Definition

- The *trace* on $S \subset \mathcal{X}$ of the ϕ-exponential family p_θ is the ϕ-exponential family

$$p_{|S,\theta}(x) = \exp_\phi\left(\sum_{j=1}^m \theta_j H_j(x) - \alpha_S(\theta)\right) p(x|S), \quad x \in S.$$

- A subset $S \subset \mathcal{X}$ is *exposed* if $S = H^{-1}(F)$ and F is a *face* of the marginal polytope. Equivalently, there exists a non-negative random variable of the form $\alpha_0 + \sum_{j=1}^d \alpha_j H_j$ whose support is S.

Note that the trace is not equal to the conditioning unless $\phi = 1$.

Extended family

Theorem

Let θ_n, $n = 1, 2, \ldots$, be a sequence of parameters such that for some non-negative probability density function q we have $\lim_{n \to \infty} p(x; \theta_n) = q(x)$.

1. *If the support of q is full, $\{q > 0\} = \mathcal{X}$, then q belongs to the ϕ-exponential family for some parameter value θ.*
2. *If the support of q is defective, then the sequence θ_n is divergent, the support is an exposed subset of $\mathcal{X}$, and q belongs to the trace of the ϕ-exponential family on the support.*
3. *Viceversa, each trace on an exposed subset is a limit of elements of the family.*

Definition

The *extended* ϕ-exponential model is the closure of the ϕ-exponential model. It is parameterized by the marginal polytope.

Part 2
Nonparametric ϕ-exponential families

- G. Pistone, C. Sempi, Ann. Statist. **23**(5), 1543 (1995), ISSN 0090-5364
- R.F. Vigelis, C.C. Cavalcante, Journal of Theoretical Probability (2011), online First

ϕ-exponential manifold I

- On a *finite* state space $(\Omega, \mathcal{F}, \mu)$, the open convex set $\mathcal{M}_>$ of positive densities is an exponential model. The intrinsic geometry of the exponential structure induces the e-geometry on $\mathcal{M}_>$ in the form of a differentiable manifold modelled on $\mathbb{R}^d$, where $d + 1 = \#\Omega$.
- On a *general* state space $(\Omega, \mathcal{F}, \mu)$, the same idea works but one has to carefully select a model Banach space for the infinite dimensional manifold supported by $\mathcal{M}_>$.
- One option is to fix a reference density p and consider the densities q of the form

  $$q = \mathrm{e}^{u - K_p(u)} \cdot p(x) = \mathrm{e}^{u - K_p(u) + \ln p(x)}$$

  where u is a random variable uniquely determined by the reference density p and by the condition $\mathrm{E}_p[u] = 0$. The centered random variable u is the nonparametric coordinate of q in the reference p.
- In the ϕ-exponential setting, we assume $\exp_\phi$ to be defined on $\mathbb{R}$, increasing and convex. There are two options.

ϕ-exponential manifold II

- At each p the model space is the Museliac-Orlicz space determined by the modular

$$v \mapsto \mathrm{E}_\mu \left[\exp_\phi\left(v + \ln_\phi(p)\right)\right].$$

- We define the tangent space T_p to be the set of such random variables which are centered with respect to the escort probability $\propto \phi(p)$.
- Consider the densities of the form

$$q = \exp_\phi\left(u - K_p(u) + \ln_\phi(p)\right), \quad u \in T_p, \quad K_p(u) \in \mathbb{R}$$

- The coordinate u is uniquely determined because

$$u_1 - K_p(u_1) + \ln_\phi(p) = u_2 - K_p(u_2) + \ln_\phi(p)$$

implies $u_1 - u_2$ constant, hence 0.

ϕ-exponential manifold III

- The cumulant function K_p satisfies

$$\ln_\phi(q) = u - K_p(u) + \ln_\phi(p)$$

hence

$$K_p(u) = \mathrm{E}_{\phi,p}\left[\ln_\phi(p) - \ln_\phi(q)\right] = \frac{\mathrm{E}_\mu\left[\phi(p)(\ln_\phi(p) - \ln_\phi(q)\right]}{\mathrm{E}_\mu\left[\phi(p)\right]}$$

- From the concavity $\ln_\phi(q) - \ln_\phi(p) \leq \frac{1}{\phi(p)}(q-p)$, hence

$$\phi(p)K_p(u) \geq \phi(p)u + p - q,$$

in particular $K_p(u) \geq 0$.

ϕ-exponential manifold IV

- Assume that $r \in \mathcal{M}_>$ is represented at p and at q:

$$
\begin{aligned}
q &= \exp_\phi\left(u - K_p(u) + \ln_\phi(p)\right), \quad u \in T_p, \\
r &= \exp_\phi\left(v - K_p(v) + \ln_\phi(p)\right), \quad v \in T_p, \\
&= \exp_\phi\left(w - K_q(w) + \ln_\phi(q)\right), \quad w \in T_q.
\end{aligned}
$$

It follows

$$v - K_p(v) = w - K_q(w) + u - K_p(u),$$

and, taking the expectation at the escort p,

$$-K_p(v) = \mathrm{E}_{\phi,p}\left[w\right] - K_q(w) - K_p(u),$$

and substracting

$$v = w - \mathrm{E}_{\phi(p)}\left[w\right] + u$$

ϕ-exponential manifold V

- Let us compute the Gateaux derivative of $u \mapsto K_p(u)$ in the direction $v \in T_p$.

$$0 = \frac{d}{d\theta} \mathrm{E}_\mu \left[\exp_\phi \left((u + \theta v) - K_p(u + \theta v) + \ln_\phi (p) \right) \right]$$
$$= \mathrm{E}_\mu \left[\phi(\exp_\phi \left((u + \theta v) - K_p(u + \theta v) + \ln_\phi (p) \right))(v - \frac{d}{d\theta} K_p(u + \theta v)) \right]$$

then

$$\frac{d}{d\theta} K_p(u + \theta v) \Big|_{\theta=0} = DK_p(u)v = \mathrm{E}_{\phi(p_u)} [v]$$

Part 3
ϕ-exponential martingales

Joint work in progress with Marina Santacroce and Barbara Trivellato (Politecnico di Torino)

- B. Trivellato, International Journal of Theoretical and Applied Finance (2012), accepted

Second derivative

Second derivative

$$\exp_\phi''(u) = (\gamma'(u) + \gamma^2(u)) \exp_\phi(u), \quad u \neq -m.$$

The second derivative exists at $-m$ if

$$\limsup_{u \downarrow -m} \gamma'(u) = 0$$

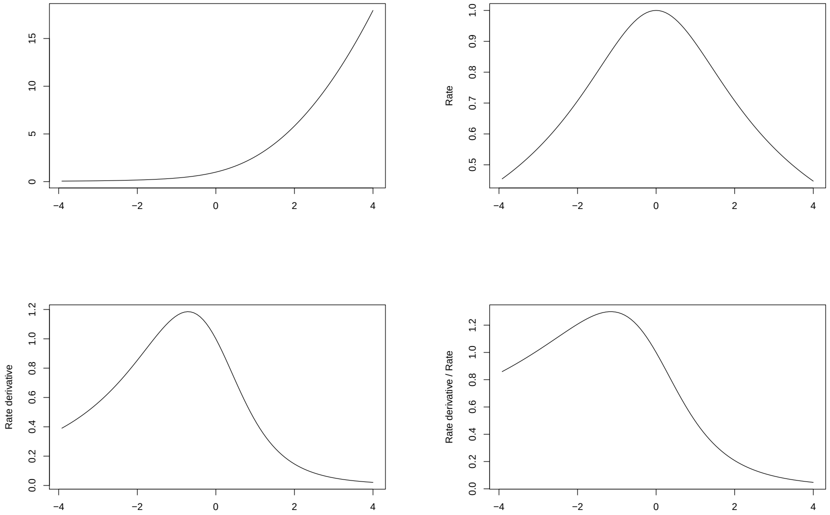
Tsallis $\exp_q$ If $q = 1/2$,

$$\exp_{1/2}(u) = \left(1 + \tfrac{1}{2}u\right)_+^2 \qquad \exp'_{1/2}(u) = \left(1 + \tfrac{1}{2}u\right)_+$$

$$\gamma(u) = \left(1 + \tfrac{1}{2}u\right)^+ \qquad \gamma'(u) = -\tfrac{1}{2}\left[\left(1 + \tfrac{1}{2}u\right)^+\right]^2$$

$$\gamma'(u)/\gamma(u) = -\tfrac{1}{2}\left(1 + \tfrac{1}{2}u\right)^+$$

Kaniadakis $\exp_\kappa$



Martingale measure and Girsanov Theorem

On the stochastic basis

$$\mathcal{B} = (\Omega, \mathcal{F}, F = (\mathcal{F}_t \colon t \in [0, T]), \mathbb{P})$$

consider a Brownian motion W and a continuous semi-martingale $X = M + A$ with quadratic variation $[X]$.

Theorem

1. *The process*

$$Z_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)$$

   is a positive local martingale such that $dZ_t = Z_t dM_t$.

2. *Z is a martingale if $\mathrm{E}(Z_T) = 1$. In such a case $\mathbb{Q} = Z_T \cdot \mathbb{P}$ is a probability measure equivalent to $\mathbb{P}$.*

3. *The process*

$$\widetilde{W}_t = W_t - [X, W]_t$$

   is a $\mathbb{Q}$-brownian motion

Proofs

1. The Ito formula for Z gives

$$dZ_t = Z_t dM_t - \frac{1}{2} Z_t d[X]_t + \frac{1}{2} Z_t d[X]_t = Z_t dM_t$$

2. Equivalence $\mathbb{P} \sim \mathbb{Q}$ follows from the strict positivity of the exponential function. Conditions on X implying $\mathrm{E}\left(Z_T\right) = 1$ are difficult because of the exponential growth.

3. Use Lévy theorem. Because of the equivalence,

$$[\widetilde{W}]_{\mathbb{Q}} = [\widetilde{W}]_{\mathbb{P}} = [W].$$

The $\mathbb{Q}$-martingale property follows from the Ito formula for the product,

$$\begin{aligned} d(Z_t \widetilde{W}_t) &= Z_t d\widetilde{W}_t + \widetilde{W}_t dZ_t + d[Z, \widetilde{W}]_t \\ &= (Z_t dW_t + \widetilde{W}_t dZ_t) + \\ &\quad (-Z_t d[M, W]_t + d[Z, W]_t), \quad d[Z, W]_t = Z_t d[M, W]_t \end{aligned}$$

Deformed exponential martingale I

- Assume $\exp_\phi$ defined on $\mathbb{R}$, strictly positive and of class C^2 e.g., Kaniadakis' $\exp_\kappa$.
- The Ito's formula applies to the semimartingale $Z = \exp_\phi(Y)$, where $Y = M - C$, M is the local martingale part of Y and C is a process with bounded variation trajectories.

$$\begin{aligned} dZ_t &= \gamma(Y_t)Z_t dY_t + \frac{1}{2}(\gamma'(Y_t) + \gamma^2(Y_t))Z_t d[Y]_t \\ &= \gamma(Y_t)Z_t dM_t + \\ &\quad Z_t\left(-\gamma(Y_t)dC_t + \frac{1}{2}(\gamma'(Y_t) + \gamma^2(Y_t))d[M]_t\right) \end{aligned}$$

- If $\gamma(Y_t)dC_t = \frac{1}{2}(\gamma'(Y_t) + \gamma^2(Y_t))d[M]_t$, then

$$dZ_t = \gamma(Y_t)Z_t dM_t$$

Deformed exponential martingale II

- The condition can be rewritten as

$$dY_t = -\frac{\gamma'(Y_t) + \gamma^2(Y_t)}{\gamma(Y_t)} d[M]_t + dM_t$$

- Let W be a browniam motion and define $\widetilde{W} = \int \theta dW - A$ where A is a bounded variation process. Let us compute the differential of $Z\widetilde{W}$:

$$\begin{aligned} d(Z_t\widetilde{W}_t) &= Z_t d\widetilde{W}_t + \widetilde{W}_t dZ_t + d[Z, \widetilde{W}]_t \\ &= (Z_t dW_t + \widetilde{W}_t dZ_t) + \\ &\quad (-Z_t dA_t + d[Z, \int \theta W]_t), \quad d[Z, W]_t = \gamma(Y_t)\theta_t Z_t d[M, W]_t. \end{aligned}$$

The martingale condition is $dA_t = \gamma(Y_t)\theta_t d[M, W]_t$.

ϕ-exponential martingale

Theorem

1. *Assume* $(\gamma' + \gamma^2)/\gamma$ *continuous, bounded and positive. Then the system of stochastic differential equations*

$$\begin{cases} dY_t = -\frac{\gamma'(Y_t)+\gamma^2(Y_t)}{\gamma(Y_t)} d[M]_t + dM_t, & Y_0 = 0, \\ dZ_t = \gamma(Y_t) Z_t dM_t, & Z_0 = 1 \end{cases}$$

   has a unique martingale solution

$$Z_t = \exp_\phi \left(Y_t - \frac{1}{2}[Y]_t \right),$$

   hence $\mathbb{Q} = Z_T \cdot \mathbb{P} \sim \mathbb{P}$.

2. *Assume* W *is Browniam motion. Then* $d\widetilde{W}_t = dW_t - \gamma(Y_t)\theta_t d[Y, W]$ *is a* $\mathbb{Q}$*-Brownian motion.*

Discussion

- The relation with the standard Doleans exponential.
- The case C^1, e.g. Tsallis is feasible because of a generalized Ito's formula.
- When $\exp_\phi$ has polynomial growth, the L^2 CAOS expansion is feasible.

Abstract

ϕ-exponential families have been defined by J. Naudts [1] and include the statistical models introduced in Physics by C. Tsallis [2] This theory presents interesting geometric features [3], such as the notion of escort probability [4]. Here we discuss how to apply the nonparametric approach we used for ordinary exponential families [5-8] to this case [9]. In particular, we consider deformed exponentials as defined by Kaniadakis [10]. Such a non parametric extension was discussed by R.F. Vigelis and C.C. Cavalcante [11]. First, we discuss the generalization of the algebra of the finite state space case and the notion of extended exponential model [12–13]. Second, we consider the relevant non parametric differential geometry. Third, we discuss the dynamic case on a Wiener space setting [14], in particular the rephrasing of Girsanov's density theorem for deformed exponentials.

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