Stochastic Processes and Calculus 2016

Ito Calculus

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References

- Basic informations of martingale are in David Williams, Probability with martingales, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991 and the monograph below.
- The presentation of the stochastic calculus based on continuous L^2 martingales parallels Robert S. Liptser and Albert N. Shiryaev, Statistics of random processes. I, expanded ed., Applications of Mathematics (New York), vol. 5, Springer-Verlag, Berlin, 2001, General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability. 1800857.
- The examples of stochastic differential equations (SDE) are taken from Steven E. Shreve, Stochastic calculus for finance. II, Springer Finance, Springer-Verlag, New York, 2004, Continuous-time models. 2057928 (2005c:91001).

Continuous L^2 martingales

Definition

A continuous process M is an L^2 martingale if

1. $\mathbb{E}\left(M_t^2\right) < +\infty$, and
2. $\mathbb{E}\left(M_t|\mathcal{F}_s\right) = M_s, \quad 0 \leq s < t.$

Theorem

Let M_n, $n = 1, 2, \ldots$ be a sequence of L^2 continuous martingales.Let T be a finite horizon and assume that the L^2 limit of $M_n(T)$ exists, i.e. there exists a random variable M such that

$$\lim_{n\to\infty} \mathbb{E}\left((M_n(T) - M)^2\right) = 0.$$

Then there exist an L^2 continuous martingale M_t, $t \in [0, T]$ such that

$$\lim_{n\to\infty} \mathbb{E}\left(sup_{t\in[o,T]}(M_n(t) - M_t)^2\right) = 0.$$

Simple processes

- W is a Brownian motion for $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$.
- An adapted process Δ is of class L^2 if

$$\mathbb{E}\left(\int_0^T \Delta^2(t)dt\right) < +\infty \quad \text{for all } T > 0.$$

- The set of adapted processes Δ of class L^2 is a vector space.
- If Y_1 is $\mathcal{F}(t_1)$ measurable and $\mathbb{E}\left(Y_1^2\right) < +\infty$, then the process $\Delta(t) = Y_1(t_1 \leq t)$ is adapted and of class L^2. A finite sum of such processes is a simple process:

$$\Delta(t) = \sum_{j=1}^{n} Y_j(t_j \leq t).$$

- All trajectory of a simple process are pure jumps and right-continuous. If we order the t_j's in increasing order, $t_1 < t_2 < \cdots < t_n$, and $t_j \leq t < t_{j+1}$, then $\Delta_t = \sum_{i=1}^{j} Y_i$.

Ito integral of the simple process $\Delta(t) = Y_1(t_1 \le t)$, $t \ge 0$

- For $\Delta(t) = Y_1(t_1 \le t)$, define the Ito integral

$$\int_0^t \Delta(s)dW(s) = \begin{cases} 0 & \text{for } t < t_1 \\ Y_1(W(t) - W(t_1)) & \text{for } t_1 \le t \end{cases}$$
$$= Y_1(W(t) - W(t \wedge t_1))$$

- The Ito integral is a continuous martingale

$$\mathbb{E}\left(\int_0^t \Delta(u)dW(u) \,\middle|\, \mathcal{F}(s) \right) = \int_0^s \Delta(u)dW(u), \quad s \le t.$$

- The Ito integral is isometric

$$\mathbb{E}\left(\left(\int_0^t \Delta(u)dW(u) \right)^2 \right) = \mathbb{E}\left(\int_0^t \Delta^2(u)du \right).$$

- The quadratic variation of the Ito integral is $\int_0^t \Delta^2(s)ds$.

Ito integral of a simple process Δ

- For $\Delta(t) = \sum_{j=1}^{n} Y_j(t_j \le t)$, define the Ito integral by linearity. If the interval $[0, t]$ contains the jumps $0 \le t_1 < \cdots t_m \le t$,

$$\begin{aligned}
\int_0^t \Delta(s) dW(s) &= \sum_{j=1}^{n} Y_j (W(t) - W(t \wedge t_j) \\
&= \sum_{j=1}^{m} Y_j (W(t) - W(t_j)) \\
&= \sum_{j=1}^{m} Y_j (\sum_{i=j+1}^{m} W(t_{i+1}) - W(t_i)) \\
&= \sum_{i=1}^{m} \Delta(t_i)(W(t_{i+1}) - W(t_i))
\end{aligned}$$

- The Ito integral is a continuous martingale.
- The Ito integral is isometric.
- The quadratic variation of the Ito integral is $\int_0^t \Delta^2(s) ds$.

Ito integral of an L^2 process

- If Δ is a process of class L^2, there exists a sequence Δ_n, $n = 1, 2, \dots$ of simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T |\Delta(u) - \Delta_n(u)|^2 \, du\right) = 0.$$

- The Ito integral of a process of class L^2 is defined by continuity.
- The Ito integral is a linear operator mapping L^2 processes into continuous martingale.
- The Ito integral is isometric.
- The quadratic variation of the Ito integral is

$$\left[\int \Delta dW\right](t) = \int_0^t \Delta^2(u) du$$

Continuous martingales

If M is a continuous bounded martingale, the computation

$$\begin{aligned} M^2(t) - M^2(s) &= \sum_{j=1}^{n} M^2(t_j) - M^2(t_{j-1}) \\ &= \sum_{j=1}^{n} 2M(t_{j-1})(M(t_j) - M(t_{j-1})) + \sum_{j=1}^{n} (M(t_j) - M(t_{j-1}))^2 \end{aligned}$$

produces the decomposition

$$M^2(t) = M^2(0) + 2\int_0^t M(u)dM(u) + [M](t)$$

and, for an Ito integral,

$$\left(\int_0^t \Delta dW\right) = 2\int_0^t \left(\int_0^s \Delta(u)dW(u)\right) dW(s) + \int_0^t \Delta^2(s)ds$$

Ito-Doeblin formula

Definition (Ito process)

An Ito process is a process of the form

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds.$$

Theorem (Ito-Doeblin formula for the Brownian Motion)

If

- $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^2)$ *and*
- $f_x(t, W(t))$, $t \geq 0$, *is an* L^2 *process,*

then $f(t, W(t))$, $t \geq 0$, *is an Ito process, and*

$$f(t, W(t)) = f(0, W(0)) + \int_0^t f_t(s, W(s))ds +$$

$$\int_0^t f_x(s, W(s))dW(s) + \frac{1}{2}\int_0^t f_{xx}(s, W(s))ds.$$

Proof of Ito-Doeblin formula I

We write for $0 \leq s < t \leq T$

$$\begin{aligned} \int_s^t \Delta(u) dW(u) &= \int_0^t \Delta(u) dW(u) - \int_0^s \Delta(u) dW(u) \\ &= \int_0^T (s < u \leq t) \Delta(u) dW(u). \end{aligned}$$

In particular,

$$\begin{aligned} (W(t) - W(s))^2 &= W(t)^2 - W(s)^2 - 2W(s)(W(t) - W(s)) \\ &= 2\int_s^t W(u) dW(u) + (t-s) - 2W(s)(W(t) - W(s)) \\ &= (t-s) + 2\int_s^t (W(u) - W(s)) dW(u) \end{aligned}$$

Proof of Ito-Doeblin formula II

The Taylor formula of order 1,2 for f gives

$$\begin{aligned}
f(t,W(t)) - f(s,W(s)) =& f_t(s,W(s))(t-s) \\
&+ f_x(s,W(s))(W(t)-W(s)) \\
&+ \frac{1}{2} f_{xx}(s,W(s))(W(t)-W(s))^2 \\
&+ R_{1,2}(s,t,W(s),W(t)) \\
=& f_t(s,W(s))(t-s) \\
&+ f_x(s,W(s))(W(t)-W(s)) \\
&+ \frac{1}{2} f_{xx}(s,W(s))(t-s) \\
&+ f_{xx}(s,W(s)) \int_s^t (W(u)-W(s))dW(u) \\
&+ R_{1,2}(s,t,W(s),W(t))
\end{aligned}$$

Summing over a partition, the first tree term go to the Ito formula, the last two terms go to zero.

Ito-Doeblin formula: Applications I

- The process $f(t, W(t))$ is a martingale if $f_{10}(t,x) + \frac{1}{2} f_{02}(t,x) = 0$.
- Let $H_n(x)$ be a polynomial of degree n and define $f(t,x) = t^{n/2} H_n(t^{-1/2}x)$. We have

$$f_{1,0}(t,x) = t^{n/2-1}(\frac{1}{2} H_n(t^{-1/2}x) - \frac{x}{2} H_n'(t^{-1/2}x)),$$
$$f_{02}(t,x) = t^{n/2-1} H_n''(t^{-1/2}x).$$

- The martingale condition is satified if

$$nH_n(y) - yH_n'(y) + H_n''(y) = 0.$$

Ito-Doeblin formula: Applications II

- We can take the Hermite polynomials

$$H_n(y) = (-1)^n \mathrm{e}^{\frac{y^2}{2}} \frac{d^n}{dy^n} \mathrm{e}^{-\frac{y^2}{2}}$$

  to obtain the Hermite martingales

$$M_n(t) = \int_0^t u^{\frac{n}{2}} H_n(u^{-\frac{1}{2}} W(u)) dW(u).$$

  [Hint: the n-th derivative of $yg(y)$ is $yg^{(n)}(y) + ng^{(n-1)}(y)$]

- As $H_n'(y) = nH_{n-1}(y)$, if $f_n(t,x) = t^{n/2} H_n(t^{-1/2}x)$, the x-derivative is

$$\frac{d}{dx} f_n(t,x) = t^{\frac{n}{2}-\frac{1}{2}} H_n'(t^{-1/2}x) = nf_{n-1}(t,x),$$

  and we have the iterated Ito integrals

$$M_n(t) = \int_0^t M_{n-1}(u) dW(u).$$

Ito processes

- For an Ito process $X(t) = X_0 + M(t) + A(t)$, $t \geq 0$, the integral is defined by approximation on simple processes.
- The M part and the A part behave differenthy when the quadratic variation is considered.
- $$X^2(t) = X^2(0) + 2\int_0^t X(s)dX(s) + [M](t) =$$

  $$X_0^2 + 2\int_0^t X(s)\Delta(s)dW(s) + 2\int_0^t X(s)\Theta(s)ds + \int_0^t \Delta^2(s)ds$$
- The quadratic variation of X and the quadratic variation of M are equal.

Ito-Doebin for Ito process

Theorem (Ito-Doeblin formula for the Ito process)

If

- $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$,
- X *is a Ito process with* $dX(t) = \Delta(t)dW(t) + \Theta(t)dt$,
- $f_x(t, X(t))\Delta(t)$, $t \geq 0$, *is an* L^2 *process,*

then $f(t, X(t))$, $t \geq 0$, *is an Ito process, and*

$$f(t, X(t)) =$$

$$f(0, X(0)) + \int_0^t f_t(s, X(s))ds + \int_0^t f_x(s, X(s))dX(s) + \frac{1}{2}\int_0^t f_{xx}(s, X(s))d[X](s)$$

$$= f(0, X(0)) + \int_0^t f_t(s, X(s))ds + \int_0^t f_x(s, X(s))\Delta(s)dW(s)$$

$$+ \int_0^t f_x(s, X(s))\Theta(s)ds + \frac{1}{2}\int_0^t f_{xx}(s, X(s))\Delta^2(s)ds$$

Geometric Brownian Motion

The process $f(t, W(t))$ is a martingale if $f_{10}(t,x) + \frac{1}{2} f_{02}(t,x) = 0$, for example

$$f(t,x) = \exp\left(\theta x - \frac{1}{2}\theta^2 t\right).$$

In such a case $f(0,0) = 1$ and

$$f_{01}(t,x) = \theta f(t,x).$$

Definition (Geometric Brownian motion)

The process $X(t) = \exp\left(\theta W(t) - \frac{1}{2}\theta^2 t\right)$ is a positive martingale and

$$X(t) = 1 + \theta \int_0^t X(u) dW(u)$$

More generally, the process $X(t) = \exp\left(\int_0^t \theta(u) dW(u) - \frac{1}{2} \in_0^t \theta^2(u) du\right)$ is a positive martingale and $dX(t) = \theta(t) X(t) dW(t)$.

Vasicek interest rate model, Example 4.4.10

The solution of the stochastic differential equation SDE

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

is an Ito process. As an equation, it has the form

$$dR(t) = -\beta R(t)dt + d(\alpha t + \sigma W(t)),$$

that is it is a linear equation $dR(t) = -\beta R(t)dt + dX(t)$, forced by the Brownian motion with drift $X(t) = \beta t + \sigma W(t)$. From the Ito formula,

$$d(\mathrm{e}^{\beta t} R(t)) = \beta \mathrm{e}^{\beta t} R(t)dt + \mathrm{e}^{\beta t} dR(t) = \mathrm{e}^{\beta t} dX(t).$$

The solution is

$$\mathrm{e}^{\beta t} R(t) = R(0) + \int_0^t \mathrm{e}^{\beta t} dX(t).$$

Cox-Ingersoll-Ross interest rate model, Example 4.4.11

The solution of the non linear SDE

$$dR(t) = (\alpha - \beta R(t))dt + \sqrt{R(t)}\sigma dW(t)$$

is an Ito process. We can write

$$dR(t) = -\beta R(t)dt + (\alpha dt + \sqrt{R(t)}\sigma dW(t)) = -\beta R(t)dt + dY(t),$$

which suggests to compute

$$\begin{aligned} d(\mathrm{e}^{\beta t}R(t)) &= \beta \mathrm{e}^{\beta t}R(t) + \mathrm{e}^{\beta t}dR(t) \\ &= \mathrm{e}^{\beta t}\alpha dt + \mathrm{e}^{\beta t}\sigma\sqrt{R(t)}dW(t). \end{aligned}$$

The expected value is computable. Same for the second moment.

Black-Scholes-Merton equation, §4.5 I

Portfolio value $X(t)$

Stock value $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$

Share $\Delta(t)$

Share value $\Delta(t)S(t)$

Differential portfolio value $dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$

We have

$$d(\mathrm{e}^{-rt}S(t)) = (\alpha - r)\mathrm{e}^{-rt}S(t)dt + \sigma\mathrm{e}^{-rt}S(t)dW(t)$$
$$d(\mathrm{e}^{-rt}X(t)) = \Delta(t)d(\mathrm{e}^{-rt}S(t))$$

Black-Scholes-Merton equation, §4.5 II

let us assume that the the call at time t is a function of stock value $S(t)$, $c(t, S(t))$ and let us compute the differential of the discounted call $\mathrm{e}^{-rt}c(t,x)$ by the Ito-Doeblin forlula. From

$$\frac{\partial}{\partial t}\mathrm{e}^{-rt}c(t,x) = \mathrm{e}^{-rt}(-rc(t,x) + c_{10}(t,x))$$
$$\frac{\partial}{\partial x}\mathrm{e}^{-rt}c(t,x) = \mathrm{e}^{-rt}c_{01}(t,x)$$
$$\frac{\partial^2}{\partial x^2}\mathrm{e}^{-rt}c(t,x) = \mathrm{e}^{-rt}c_{02}(t,x)$$

we obtain

$$\begin{aligned} d(\mathrm{e}^{-rt}c(t,S(t))) =& \mathrm{e}^{-rt}(-rc(t,S(t)) + c_{10}(t,S(t)))dt \\ &+ \mathrm{e}^{-rt}c_{01}(t,S(t))dS(t) + \frac{1}{2}\mathrm{e}^{-rt}c_{02}(t,S(t))d[S](t) \end{aligned}$$

Black-Scholes-Merton equation, §4.5 III

and, substituting the differentials

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) \\ d[S](t) &= \sigma^2 S^2(t)dt \end{aligned}$$

we get

$$\begin{aligned} d(\mathrm{e}^{-rt}c(t,S(t))) =& \mathrm{e}^{-rt}(-rc(t,S(t)) + c_{10}(t,S(t)))dt \\ &+ \mathrm{e}^{-rt}c_{01}(t,S(t))(\alpha S(t)dt + \sigma S(t)dW(t)) \\ &+ \frac{1}{2}\mathrm{e}^{-rt}c_{02}(t,S(t))\sigma^2 S^2(t)dt \end{aligned}$$

Now we look for an equation for $c(t,x)$ such that

$$d(\mathrm{e}^{-rt}X(t)) = d(\mathrm{e}^{-rt}c(t,S(t)))$$

We are comparing two Ito process. Forst we equate the martingale terms

$$\mathrm{e}^{-rt}c_{01}(t,S(t))\sigma S(t)dW(t) = \mathrm{e}^{-rt}\Delta(t)\sigma S(t)dW(t).$$

Black-Scholes-Merton equation, §4.5 IV

The equality is true if

$$\Delta(t) = c_{01}(t, S(t)).$$

$$\mathrm{e}^{-rt} c_{01}(t, S(t))(\alpha - r)S(t)dt =$$
$$\mathrm{e}^{-rt}(-rc(t, S(t)) + c_{10}(t, S(t)) + c_{01}(t, S(t))\alpha S(t) + c_{02}(t, S(t))\sigma^2 S^2(t))dt$$

The equality follows if $c(t, x)$ satisfies the BSM equation

$$\left(\frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial}{\partial x}\right) c(t, x) = rc(t, x), \quad t \in [0, T], x \geq 0,$$

together with a suitable boudary condition e.g.,

$$c(T, x) = (x - K)^+.$$