STOCHASTIC PROCESSES AND CALCULUS 2016 - 3rd week

May 2016

Martingales

- Refer to D. Williams *Probability with Martingales* Part C.
- Two martingale theorem are used in the basic construction of Wiener integrals:
- The Lévy's convergence theorem of $X_n = \mathbb{E}(X|\mathcal{F}_n)$ to $\mathbb{E}(X|\mathcal{F}_\infty)$ is discussed in §14.2. Note that $X \in L^2$ implies that the convergence holds a.s. and in L^2; this can be used to derive the simple approximation of the integrands.
- The Doob's maximal theorem $\mathbb{E}(\sup_{k\leq n} X_k^2) \leq 4\mathbb{E}(X_n^2)$ is discussed in §14.11. If X_k is a square-integrable martingale, then X_k^2 is a *sub-martingale*. This is used to prove the existence of an integral with continuous trajectories.

Homeworks due May 23

1. If $f \in L^2([0,1]) \cap C([0,1])$, then

$$\lim \sum_j f(t_{j_1})(W_{t_j} - W_{t_{j-1}}) = \int f(t)\ dW_t,$$

where the limit is taken along any sequence of partition of $[0,1]$ such that $\max(t_j - t_{j_1}) \to 0$.

Hint. Show that the $L^2(0,1)$ norm of the difference between the left-of-the-interval approximation and the mean approximation on the interval of such a function goes to zero.

2 If $f \in L^2([0,1]) \cap C^1([0,1])$, then

$$\int_0^1 f(u)\ dW_u = f(1)W_1 - f(0)W_0 - \int_0^1 f'(u)W_u\ du.$$

Hint. Write

$$\begin{aligned}
f(1)W_1 - f(0)W_0 &= \sum_{j=1}^{n}(f(t_j)W_{t_j} - f(t_{j-1})W_{t_{j-1}}) \\
&= \sum_{j=1}^{n}(f(t_j)W_{t_j} - f(t_j)W_{t_{j-1}}) + \sum_{j=1}^{n}(f(t_j)W_{t_{j-1}} - f(t_{j-1})W_{t_{j-1}}) \\
&= \sum_{j=1}^{n} f(t_j)(W_{t_j} - W_{t_{j-1}}) + \sum_{j=1}^{n}(f(t_j) - f(t_{j-1}))W_{t_{j-1}} \\
&= \sum_{j=1}^{n} f(t_j)(W_{t_j} - W_{t_{j-1}}) + \sum_{j=1}^{n} f'(t_j^*)W_{t_{j-1}}(t_j - t_{j-1}) \ ,
\end{aligned}$$

with $t_{j-1} < t_j^* < t_j$.