STOCHASTIC PROCESSES AND CALCULUS 2016

1. ONE-DIMENSIONAL GAUSSIAN SPACE

GIOVANNI PISTONE

Contents

References

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M95-V: Ch V of Paul Malliavin, *Integration and probability*, Graduate Texts in Mathematics, vol. 157, Springer-Verlag, New York, 1995, With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky. MR MR1335234 (97f:28001a).

NP12-1: Ch 1 of Ivan Nourdin and Giovanni Peccati, *Normal approximations with Malliavin calculus*, Cambridge Tracts in Mathematics, vol. 192, Cambridge University Press, Cambridge, 2012, From Stein's method to universality. MR 2962301.

1. The 1-d Gauss-Sobolev space

Definition 1. If $d\gamma(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2}\, dz$ is the standard normal distribution, a 1-dimensional *Gaussian space* is a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with a random variable $Z \colon \Omega \to \mathbb{R}$, such that $\mathcal{F} = \sigma(Z)$, $Z \sim \mathrm{N}(0,1)$. On a Gaussian space every random variable Y is of the form $Y = f(Z)$.

We shall study the Hilbert space $\mathcal{H} = L^2(\mathcal{F})$, that is $Y = f(Z) \in \mathcal{H}$ means $f \in L^2(\gamma)$. The scalar product of $X = f(Z)$ and $Y = g(Z)$ in $\mathcal{H}$ is

$$\langle X, Y \rangle = \int f(z) g(z)\, \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2}\, dz = \langle f, g \rangle_\gamma \ .$$

Note that for each $X = f(Z) \in \mathcal{H}$, $\mathrm{E}\left(X\right) = \int f(z)\, dz = \langle 1, X \rangle$.

Let f be a real function, differentiable and with compact support. Then the integration by parts formula gives a remarkable result:

$$\int f'(z)\, \gamma(dz) = \int z f(z)\, \gamma(dz) \ .$$

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that generalises to the following proposition.

Proposition 1.

(1) *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be absolutely continuous, that is*

$$f(x) = f(0) + \int_0^x f'(t)\ dt$$

for some f' *integrable on every real interval. Then* f *is continuous. Moreover, if* $f' \in L^1(\gamma)$*, then* $(x \mapsto xf(x)) \in L^1(\gamma)$ *and*

$$\int zf(z)\ d\gamma(z) = \int f'(z)\ d\gamma(z)\ .$$

(2) *The previous equality applies to each polynomial. In particular, for each* $n \geq 0$

$$\int z^{2n}\ d\gamma(z) = \int z \cdot z^{2n-1}\ d\gamma(z) = (2n-1) \int z^{2(n-1)}\ d\gamma(z)\ .$$

It follows $\mathrm{E}\left(Z^{2n}\right) = (2n-1)!!$ *so that* $f(Z) \in \mathcal{H}$ *for all polynomial* f*.*

Proof. (1) See NP12-1 p. 5.
(2) The previous item and induction.

□

We now extend the previous proposition in a functional way. Note the peculiar terminology. Below, the name $\mathcal{S}$ is used in a sense which is different from its use in Fourier analysis. The name *divergence* is intended to recall that for each $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$, with proper assumptions, we have $\operatorname{grad} f(x) = (\partial_i f(x)\colon i = 1\ldots, n)$ and

$$\int \operatorname{grad} f(x) \cdot g(x)\ dx = -\int f(x) \operatorname{div} g(x)\ dx\ ,$$

hence $\operatorname{div} g(x) = \sum_{i=1}^n \partial_i g_i(x)$.

Definition 2. The function f belongs to $\mathcal{S}$ if $f \in C^\infty(\mathbb{R})$ and for each derivative $f^{(n)}$ there exists a monomial x^m such that $\lim_{x\to\pm\infty} x^{-m} f^{(n)} = 0$.

Proposition 2.

(1) $\mathcal{S}$ *is an algebra with unity that contains the algebra of real polynomial functions. The derivation acts* $d\colon \mathcal{S} \to \mathcal{S}$*.*

(2) *Let* $f, g \in \mathcal{S}$*. The operator* $\delta\colon \mathcal{S}$ *defined by* $\delta f(x) = xf(x) - f'(x)$ *is called* divergence operator*. It takes values in* $\mathcal{S}$ *and*

$$\langle df, g\rangle_\gamma = \langle f, \delta g\rangle_\gamma$$

(3) *On* $\mathcal{S}$*,*

$$d\delta - \delta d = I$$

Proof. (1) Let ϕ be the Lebesgue density of γ. If $f \in \mathcal{S}$, then $\frac{d}{d[} f(x)(g(x)\phi(x))] = f'(x)g(x)\phi(x) + f(x)\delta g(x)\phi(x)$ and each term is Lebesgue integrable.
(2) A computation.

□

Proposition 3. *Let* $\mathcal{S} \circ Z$ *be the vector subspace of* $\mathcal{H}$ *of random variables* X *such that* $X = f(z)$ *with* $f \in \mathcal{S}$*. It is a dense subspace of* $\mathcal{H}$*.*

Proof. There is a proof in NP12-1 p. 6 based on Functional Analysis and Fourier Analysis. A different proof could be based on a variant of the Monotone Class Theorem of W91¶3.14. In fact, the property of being a π-system for a family of events becomes the property of being closed for the product in the case of a family of real functions. Precisely, if $f_n, g_n \in \mathcal{S}$ and $f_n \downarrow \mathbf{1}_A$ and $g_n \downarrow \mathbf{1}_B$, $n \to \infty$ and $A, B \in \mathcal{F}$, then $f_n g_n \in \mathcal{S}$ and $f_n g_n \downarrow \mathbf{1}_{A\cap B}$. ☐

Proposition 4. (1) *The derivation operator* $d\colon \mathcal{S} \to L^2(\gamma)$ *is* closable*, that is if the sequence* $(f_n)_{n\in\mathbb{Z}}$ *of functions in* $\mathcal{S}$ *converges to 0 in* $L^2(\gamma)$ *and the sequence of derivatives* $(f_n')_{n\in\mathbb{Z}}$ *converges to* $\eta \in L^2(\gamma)$*, then* $\eta = 0$.

(2) *The* Gauss-Sobolev space $\mathbb{D}$ *of real functions which are absolutely continuous and such that* $f, f' \in L^2(\gamma)$ *with the* graph norm

$$\|f\|_{\mathbb{D}} = \left(\|f\|_\gamma^2 + \|f'\|_\gamma^2\right)$$

is an Hilbert space, with $\mathcal{S} \subset \mathbb{D} \hookrightarrow L^2(\gamma)$.

(3) $\mathcal{S}$ *is dense in* $\mathbb{D}$*,* d *extends to* $\mathbb{D}$ *as* $df = f'$ *and* δ *extends to a domain containing* $\mathbb{D}$ *with*

$$\langle df, g\rangle_\gamma = \langle f, \delta g\rangle_\gamma\,, \quad f \in \mathbb{D}, g \in \operatorname{Dom}\delta\ .$$

Proof. (1) Cf. NP12-1 p. 7, where a more general case is discussed. For $g \in \mathcal{S}$ we have

$$\begin{aligned}\int \eta(x)g(x)\ \gamma(dx) &= \lim_{n\to\infty}\int df_n(x)g(x)\ \gamma(dx) \\ &= \lim_{n\to\infty}\int f_n(x)\delta g(x)\ \gamma(dx) = 0\ .\end{aligned}$$

(2) It is a general argument of Functional Analysis. Essentially, it is the same argument we have above, with f' in place of df.

(3) Again, it is a check of the definitions. Note that the extension is already given and we need only to consider the definition of δ on $\operatorname{Dom}\delta$ by a standard argument in Functional Analysis.

☐

Remark 1. The space $\mathbb{D}$ has 1 derivative and square integrability. Wher more generality is allowed for, it should be denoted by $\mathbb{D}^{1,2}$. With this notation, for example, the space $\mathbb{D}^{2,2}$ has derivatives up to the secon order, that is $f'(x) = f'(0) + \int_0^x f''(u)\ du$ and

$$f(y) = f(0) + f'(0)y + \int_0^y (y-u)f''(u)\ du\ .$$

However, this simple setting is not available when the number of independent gaussian in the Gaussian space is larger than 1.

2. Hermite polynomials

In this Section we discuss an orthonormal base of $L^2(\gamma)$.

Definition 3. The 1-dimensional Hermite polynomials H_n are defined by successive application of the divergence operator to the constant function 1:

$$H_0(x) = 1, \quad H_{n+1} = \delta H_n(x)$$

Proposition 5.

(1) *Each Hermite polynomial* H_n *is a monic polynomial of degree* n.

(2) *The sequence of Hemite polynomials is total in $L^2(\gamma)$. In other words, the vector space generated by the Hermite polynomials is the ring of polynomials and it is dense in $L^2(\gamma)$.*
(3) $dH_n = nH_{n-1}$
(4) $(d+\delta)H_n(x) = xH_n(x)$
(5) $\delta d H_n = nH_n$
(6) $H_n(x) = (-1)^n \mathrm{e}^{x^2/2} \frac{d^n}{dx^n} \mathrm{e}^{-x^2/2}$
(7) *The sequence $(n!)^{-\frac{1}{2}} H_n$ is an orthonormal sequence in $L^2(\gamma)$. Because of 2 it is an orthonormal basis.*
(8) *If f is a polynomial of degree less that $2n$, then the unique polynomial r of degree less that n such that $f(x) = q(x)H_n(x) + r(x)$ (division of polynomials) is such that $\mathrm{E}(f(Z)) = \mathrm{E}(r(Z))$.*
(9) *If $f \in C^\infty(\mathbb{R})$ and $f^{(n)} \in L^2(\gamma)$ for all n, then*

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \int f^{(n)}(x)\ \gamma(dx)\ H_n$$

in $L^2(\gamma)$.
(10) *In particular,*

$$\left(x \mapsto \mathrm{e}^{cx - \frac{c^2}{2}}\right) = \sum_{n=0}^{\infty} \left(x \mapsto \frac{c^n}{n!} H_n(x)\right)$$

Proof. (1) By induction.
(2) Because of the density of the set of polynomials.
(3) By induction using Prop. 2(3):

$$dH_{n+1}(x) = d\delta H_n(x) = H_n(x) + \delta d H_n(x) = H_n(x) + n\delta H_{n-1}(x) = (n+1)H_n(x)\ .$$

(4) Computation from (3).
(5) Computation from (3).
(6) By induction.
(7) Assume $m \le n$ and compute

$$\langle H_m, H_n\rangle_\gamma = \langle H_m, \delta^n 1\rangle_\gamma = \langle d^n H_m, 1\rangle_\gamma\ .$$

(8) From (7) we get $\int q(x)H_n(x)\ \gamma(dx) = 0$.
(9) The Fourier expansion of $f \in L^2(\gamma)$ is

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \langle f, H_n\rangle_\gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \langle f, \delta^n 1\rangle_\gamma$$

and, by induction, $\langle f, \delta^n 1\rangle_\gamma = \langle d^n f, 1\rangle_\gamma = \int f^{(n)}(x)\ \gamma(dx)$.

□

Proposition 6.
(1) *For all $n = 1, 2, \dots$ the Hermite polynomial $H_n(x)$ has n real roots which are separated by the $n-1$ real roots of $H_{n-1}(x)$*
(2) *There exists weights $w_1, \dots, w_n \in \mathbb{R}_+$ such that for each polynomial of degree less than $2n-1$*

$$\mathrm{E}(f(Z)) = \sum_{j=1}^{n} w_j f(x_j)\ .$$

Proof. (1) We proceed by induction. The theorem is true for the couple $H_1(x)$, $H_2(x)$. From $H'_{n+1}(x) = (n+1)H_n(x)$ we see that the derivative of $H_{n+1}(x)$ is zero at each of the n roots of $H_n(x)$. Moreover, at the same points, the second derivative is $H''_{n+1}(x) = (n+1)nH_{n-1}(x)$ and cannot be zero by the induction assumption.

(2) First reduce to the case of degree less than n with Prop. 5(8). Let $x_1, \dots, x_n$ be the roots of $H_n(x)$ and let $L_{n,k}(x)$, $k = 1, \dots, n$ be the Lagrange polynomials. for each real function g, the polynomial $L_n[g](x) = \sum_{k=1}^n g(x_k)L_{n,k}(x)$ interpolates g and it is equal everywhere if g is a polynomial of degree less than n. In conclusion

$$\mathrm{E}\left(f(Z)\right) = \mathrm{E}\left(\sum_{k=1}^n f(x_k)L_{n,k}(Z)\right) = \sum_{k=1}^n f(x_k)\,\mathrm{E}\left(L_{n,k}(Z)\right) \ .$$

$\square$

Definition 4.

(1) For each $\boldsymbol{u} \in S_2 = \{\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2 | u_1^2 + u_2^2 = 1\}$, define the operator $Q_{\boldsymbol{u}} \colon \mathcal{S}$ by

$$Q_{\boldsymbol{u}} f(x) = \int f(u_2 x + u_1 z)\ \gamma(dz) = \mathrm{E}\left(f(u_2 x + u_1 Z)\right)$$

(2) For each $t \geq 0$ the *Ornstein-Uhlenbeck semi-group* is the second order differential operator defined on $\mathcal{S}$ by

$$P_t f(x) = Q_{(\sqrt{1-\mathrm{e}^{-2t}}, \mathrm{e}^{-t})} f(x).$$

Here *semi-group* means that $P_o f = f$ and $P_{s+t} f = P_s P_t f$.

(3) The *Ornstein-Uhlenbeck operator* $\mathcal{L}$ is defined on $\mathcal{S}$ by

$$\mathcal{L} f(x) = \delta d f(x) = -\frac{d^2}{dx^2} f(x) + x \frac{d}{dx} f(x)$$

(4) The Hermite polynomials are the *eigen-functions* of $\mathcal{L}$:

$$\mathcal{L} H_n(x) = n H_n(x)$$

We need to check the semi-group property.

Proposition 7.

(1) $Q_{\boldsymbol{u}}$, *hence* P_t *extend to contraction operators on* $L^2(\gamma)$.
(2) $\langle Q_{\boldsymbol{u}} f, g\rangle_\gamma = \langle f, Q_{\boldsymbol{u}} g\rangle_\gamma$ *that is both* $Q_{\boldsymbol{u}}$ *and* P_t *are auto-adjoint.*
(3) $dQ_{\boldsymbol{u}} f(x) = u_2 Q_{\boldsymbol{u}}(df)(x)$ *if* $f \in \mathbb{D}$.
(4) $Q_{\boldsymbol{u}}(\delta f)(x) = u_2 \delta Q_{\boldsymbol{u}} f(x)$
(5) $\mathcal{L} Q_{\boldsymbol{u}} = Q_{\boldsymbol{u}} \mathcal{L}$
(6) $Q_{\boldsymbol{u}} H_n = u_2^n H_n$

Proof.

(1) It is a direct check using the fact $u_1 Z_1 + u_2 Z_2 \sim \mathrm{N}(0,1)$ if Z_1, Z_2 are independent and $\mathrm{N}(0,1)$.
(2) Use the rotational invariance of the distribution $\gamma \otimes \gamma$.

$\square$

Proposition 8.

(1) $P_0 f = f$.
(2) $P_s \circ P_t = P_{s+t}$.
(3) $\frac{d}{dt} P_t f = -\mathcal{L} P_t f$, *in particular* $\left.\frac{d}{dt} P_t f\right|_{t=0} = -\mathcal{L} f$.

Proposition 9 (Poincaré inequality)**.** *If $Z \sim N(0,1)$ and $f \in \mathbb{D}^{1,2}$, then*

$$\operatorname{Var}(f(Z)) \le \operatorname{E}\left(f'(Z)^2\right)$$

Proof. See [NP12-1] p. 12.

$$\begin{aligned}
\operatorname{Var}(f(Z)) &= \operatorname{E}(f(Z)(f(Z) - \operatorname{E}(f(Z)))) \\
&= \operatorname{E}(f(Z)(P_0 f(Z) - P_\infty f(Z))) \\
&= -\int_0^\infty \operatorname{E}\left(f(Z)\frac{d}{dt}P_t f(Z)\right)\, dt \\
&= \int_0^\infty \operatorname{E}(f(Z)\delta D P_t f(Z))\, dt \\
&= \int_0^\infty \operatorname{E}(df(Z) D P_t f(Z))\, dt \\
&= \int_0^\infty \mathrm{e}^{-t}\operatorname{E}(df(Z) P_t df(Z))\, dt \\
&\le \int_0^\infty \mathrm{e}^{-t}\sqrt{\operatorname{E}(df(Z)^2)}\sqrt{\operatorname{E}(P_t df(Z)^2)}\, dt \\
&\le \int_0^\infty \mathrm{e}^{-t}\operatorname{E}\left(df(Z)^2\right)\, dt = \operatorname{E}\left(df(Z)^2\right)
\end{aligned}$$

□

Proposition 10 (Variance expansion)**.** *If $Z \sim N(0,1)$ and $f \in \mathcal{S}$, then*

$$\operatorname{Var}(f(Z)) = \sum_{n=0}^\infty \frac{1}{n!}\operatorname{E}\left(f^{(n)}(Z)\right)^2.$$

If, moreover $\operatorname{E}\left(f^{(n)}(Z)^2\right)/n! \to 0$, *then*

$$\operatorname{Var}(f(Z)) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!}\operatorname{E}\left(f^{(n)}(Z)^2\right).$$

Proof. See [NP12-1] pp. 15-16 The Fourier expansion of f is

$$f(Z) - \operatorname{E}(f(Z)) = \sum_{n=1}^\infty \frac{1}{n!}\operatorname{E}\left(f^{(n)}(Z)\right) H_n(Z)$$

and

$$\operatorname{E}\left((f(Z) - \operatorname{E}(f(Z)))^2\right) = \sum_{n=1}^\infty \left(\frac{1}{\sqrt{n!}}\operatorname{E}\left(f^{(n)}(Z)\right)\right)^2$$

which proves the first part. For the second part, define

$$g(t) = \operatorname{E}\left(Q_{(\sqrt{1-t},\sqrt{t})} f(Z)^2\right), \quad 0 \le t \le 1$$

and note that $\operatorname{Var}(f(Z)) = g(1) - g(0)$. Then compute the Taylor expansion. □

Home work

Solve two of the following exercises. The paper is due on Mon May 9.

Exercise 1. Compute the Fourier transform of each H_n.

Exercise 2. Does $x \mapsto |x|$ belong to $\mathbb{D}$? Prove or dispove.

Exercise 3. Let ϕ be the Gaussian $\mathrm{N}(0,1)$ density. Then $\phi_a = y \mapsto a^{-1}\phi(a^{-1}y)$ is the $\mathrm{N}(0,a^2)$ density. For each $f \in L^2(\gamma)$ define

$$f_a(y) = f \star \phi_a(y) = \int f(x)\phi_a(y-x)\ dx = \int f(y-x)\phi_a(x)\ dx$$

Is $f_a \in \mathcal{S}$? If $f_a \in \mathcal{S}$, compute the derivatives. Does f_a converge to f in $L^2(\gamma)$, $a \to 0$?. If $f \in \mathcal{D}$ does $f_a \to f$ in $\mathbb{D}$?

Exercise 4. Provide a numerical application of Proposition 6, e.g. compute $\mathrm{E}\left(\cos(X)^2\right)$.

As a second option, chose one among the exercises in Sec. 1.7 of the book [NP12-1].

Collegio Carlo Alberto
E-mail address: `giovanni.pistone@carloalberto.org`