STOCHASTIC PROCESSES 2015

1. ONE-DIMENSIONAL GAUSSIAN SPACE

GIOVANNI PISTONE

Contents

References

M95-V: See Ch V of Paul Malliavin, *Integration and probability*, Graduate Texts in Mathematics, vol. 157, Springer-Verlag, New York, 1995, With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky. MR MR1335234 (97f:28001a)

NP12-1: See Ch 1 of Ivan Nourdin and Giovanni Peccati, *Normal approximations with Malliavin calculus*, Cambridge Tracts in Mathematics, vol. 192, Cambridge University Press, Cambridge, 2012, From Stein's method to universality. MR 2962301

1. 1-dimensional Gaussian space

Definition 1. If $d\gamma(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2}\ dz$ is the standard normal distribution, a 1-dimensional *Gaussian space* is a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with a random variable $Z \colon \Omega \to \mathbb{R}$, such that $\mathcal{F} = \sigma(Z)$, $Z \sim \mathrm{N}(0,1)$. On a Gaussian space every random variable Y is of the form $Y = f(Z)$.

Proposition 1.

(1) *Let $f \colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous, that is*

$$f(x) = f(0) + \int_0^x f'(t)\ dt$$

for some f' integrable on every real interval. Then f is continuous. Moreover, if $f' \in L^1(\gamma)$, then $(x \mapsto x f(x)) \in L^1(\gamma)$ and

$$\int z f(z)\ d\gamma(z) = \int f'(z)\ d\gamma(z)$$

Date: Moncalieri, May 8, 2015.

(2) *The previous equality applies to each polynomial. In particular, for each $n \geq 0$*

$$\int z^{n+2} \, d\gamma(z) = \int z \cdot z^{n+1} \, d\gamma(z) = (n+1) \int z^n \, d\gamma(z)$$

Definition 2. The function f belongs to $\mathcal{S}$ if $f \in C^\infty(\mathbb{R})$ and for each derivative $f^{(n)}$ there exists a monomial x^m such that $\lim_{x \to \pm\infty} x^{-m} f^{(n)} = 0$.

Proposition 2.

(1) *Let $f, g \in \mathcal{S}$. The operator $\delta \colon \mathcal{S}$ defined by $\delta f(x) = x f(x) - f'(x)$ is called* divergence operator. *It takes values in $\mathcal{S}$ and*

$$\langle df, g \rangle_\gamma = \langle f, \delta g \rangle_\gamma$$

(2) *On $\mathcal{S}$,*

$$d\delta - \delta d = I$$

Remark 1. Consider the space $W^{1,1}(\mathbb{R})$ of functions which are absolutely continuous and such that $f, f' \in L^1(\gamma)$. Both the operator d and δ are well defined on $W^{1,1}(\mathbb{R})$ and this definition extends the definition on $\mathcal{S}$. It is of interest the following property. Let $(f_n)_n$ be a sequence in $W^{1,1}(\mathbb{R})$ such that $f_n \to 0$ and $f'_n \to \eta$ in $L^1(\mathbb{R})$. If this implies $\eta = 0$ we say that the operator d is *closed* and, in turn, $W^{1,1}$ is a Banach space.

2. Hermite polynomials

Definition 3. The 1-dimensional Hermite polynomials H_n are defined by successive application of the divergence operator to the constant function 1:

$$H_0(x) = 1, H_{n+1} = \delta H_n(x)$$

Proposition 3.

(1) *Each Hermite polynomial H_n is a monic polynomial of degree n.*
(2) *If f is a polynomial of degree less that $2n$, then the unique polynomial r of degree less that n such that $f(x) = q(x) H_n(x) + r()$ is such that $\mathrm{E}\left(f(Z)\right) = \mathrm{E}\left(r(Z)\right)$*
(3) *The sequence of Hemite polynomials is total in $L^2(\gamma)$. In other words, the vector space generated by the Hermite polynomials is the ring of polynomials and it is dense in $L^2(\gamma)$.*
(4) *The sequence $(n!)^{-\frac{1}{2}} H_n$ is an orthonormal sequence in $L^2(\gamma)$. Because of 3 it is an orthonormal basis.*
(5) *If $f \in C^\infty(\mathbb{R})$ and $f^{(n)} \in L^2(\gamma)$ for all n, then*

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \int f^{(n)} \, d\gamma(z) H_n$$

in $L^2(\gamma)$.
(6) *In particular,*

$$\left(x \mapsto \mathrm{e}^{cx - \frac{c^2}{2}}\right) = \sum_{n=0}^{\infty} \left(x \mapsto \frac{c^n}{n!} H_n(x)\right)$$

(7) $dH_n = n H_{n-1}$
(8) $(d + \delta) H_n(x) = x H_n(x)$
(9) $\delta d H_n = n H_n$
(10) $H_n(x) = (-1)^n \mathrm{e}^{x^2/2} \frac{d^n}{dx^n} \mathrm{e}^{-x^2/2}$

Proposition 4.

(1) *Each Hermite polynomial has real roots only, that is, for each $n \geq 1$ there exist reals $x_1, \dots, x_n$ such that $H_n(x) = 0$ if and only if $x = x_j$ for some j*
(2) *There exists weights $w_1, \dots, w_n \in \mathbb{R}_+$ such that for each polynomial of degree less than n*

$$\mathrm{E}\left(f(Z)\right) = \sum_{j=1}^{n} w_j f(x_j)$$

Proof. This topic goes under the heading *Gaussian quadrature formulæ.* □

3. Derivation operator and divergence operator

Proposition 5.

(1) *$\mathcal{S}$ is dense in $L^2(\gamma)$, so that $Df = f'$ and $\delta f = (x \mapsto xf(x) - f'(x))$ are densely defined operators of $L^2(\gamma)$.*
(2) *The operator $D : \mathrm{Dom}\,(D) \subset L^2(\gamma) \to L^2(\gamma)$ is closable.*
(3) *The operator δ has a closed extension when defined as the adjoint of D.*

Proof.

(1) Let ϕ be the Gaussian $\mathrm{N}(0,1)$ density. Then $\phi_a = y \mapsto a^{-1}\phi(a^{-1}y)$ is the $\mathrm{N}(0, a^2)$ density. For each $f \in L^2(\gamma)$ define

$$f_a(y) = f \star \phi_a(y) = \int f(x)\phi_a(y-x)\ dx = \int f(y-x)\phi_a(x)\ dx$$

The n-th derivative of $y \mapsto \phi_a(y-x)$ is

$$\begin{aligned}\frac{d^n}{dy^n}\phi_a(y-x) &= a^{-(1+n)}\phi^{(n)}\left(a^{-1}(y-x)\right) \\ &= a^{-(1+n)}H_n(a^{-1}(y-x))\phi\left(a^{-1}(y-x)\right)\end{aligned}$$

It follows that f_a is n-differentiable for all n with

$$\begin{aligned}\frac{d^n}{dy^n}f_a(y) &= \int f(x)\frac{d^n}{dy^n}\phi_a(y-x)\ dx \\ &= a^{-(1+n)}\int f(x)H_n(a^{-1}(y-x))\phi\left(a^{-1}(y-x)\right)\ dx \\ &= a^{-n}\int f(a(z+y))H_n(z)\phi(z)\ dz\end{aligned}$$

See [NP12-1]
See [NP12-1] □

Definition 4.

(1) The set of f absolutely continuous and such that $f, f', (x \mapsto xf) \in L^2(gamma)$ is denoted by $\mathbb{D}^{1,2}$.
(2) The closure of the derivative is defined on $\mathbb{D}^{1,2}$.
(3) $\delta \colon \mathbb{D}^{1,2} \to L^2(\gamma)$ is called the *divergence operator*. It is a closable operator.

4. Ornstein-Uhlenbeck semigroup and operator

Definition 5.

(1) For each $\boldsymbol{u} \in S_2 = setof \boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2 u_1^2 + u_2^2 = 1$, define the operators $Q_{\boldsymbol{u}} \colon \mathcal{S}$ by

$$Q_{\boldsymbol{u}} f(x) = \int f(u_2 x + u_1 z)\ \varphi(z) dz = \mathrm{E}\left(f(u_2 x + u_1 Z)\right)$$

(2) For each $t \geq 0$ the *Ornstein-Uhlembeck semigroup* is defined on $\mathcal{S}$ by

$$P_t f(x) = Q_{(\sqrt{1-\mathrm{e}^{-2t}}, \mathrm{e}^{-t})} f(x).$$

Here *semigroup* means that $P_o f = f$ and $P_{s+t} f = P_s P_t f$.

(3) The *Orstein-Uhlenbeck operator* $\mathcal{L}$ is defined on $\mathcal{S}$ by

$$\mathcal{L} f(x) = \delta d f(x) = -\frac{d^2}{dx^2} f(x) + x \frac{d}{dx} f(x)$$

It follows that the Hermite polynomials are the *eigenfunctions* of $\mathcal{L}$:

$$\mathcal{L} H_n(x) = n H_n(x)$$

Proposition 6.

(1) *$Q_{\boldsymbol{u}}$, hence P_t extend to contraction operators on $L^2(\gamma)$.*
(2) *$\langle Q_{\boldsymbol{u}} f, g\rangle_\gamma = \langle f, Q_{\boldsymbol{u}} g\rangle_\gamma$ that is both $Q_{\boldsymbol{u}}$ and P_t are autoadjoint.*
(3) *$d Q_{\boldsymbol{u}} f(x) = u_2 Q_{\boldsymbol{u}}(df)(x)$ if $f \in \mathbb{D}^{1,2}$.*
(4) *$Q_{\boldsymbol{u}}(\delta f)(x) = u_2 \delta Q_{\boldsymbol{u}} f(x)$*
(5) *$\mathcal{L} Q_{\boldsymbol{u}} = Q_{\boldsymbol{u}} \mathcal{L}$*
(6) *$Q_{\boldsymbol{u}} H_n = u_2^n H_n$*

Proof.

(1) It is a direct check using the fact $u_1 Z_1 + u_2 Z_2 \sim \mathrm{N}(0,1)$ if Z_1, Z_2 are independent and $\mathrm{N}(0,1)$.
(2) Use the rotational invariance of the distribution $\gamma \otimes \gamma$.

□

Proposition 7.

(1) *$P_0 f = f$.*
(2) *$P_s \circ P_t = P_{s+t}$.*
(3) *$\frac{d}{dt} P_t f = -\mathcal{L} P_t f$, in particular $\left.\frac{d}{dt} P_t f\right|_{t=0} = -\mathcal{L} f$.*

5. Applications

Proposition 8 (Poincaré inequality)**.** *If $Z \sim N(0,1)$ and $f \in \mathbb{D}^{1,2}$, then*

$$\mathrm{Var}\left(f(Z)\right) \leq \mathrm{E}\left(f'(Z)^2\right)$$

Proof. See [NP12-1] p. 12.

$$\begin{aligned}
\mathrm{Var}\,(f(Z)) &= \mathrm{E}\,(f(Z)(f(Z)-\mathrm{E}\,(f(Z)))) \\
&= \mathrm{E}\,(f(Z)(P_0 f(Z) - P_\infty f(Z))) \\
&= -\int_0^\infty \mathrm{E}\left(f(Z)\frac{d}{dt}P_t f(Z)\right)\, dt \\
&= \int_0^\infty \mathrm{E}\,(f(Z)\delta D P_t f(Z))\, dt \\
&= \int_0^\infty \mathrm{E}\,(df(Z) D P_t f(Z))\, dt \\
&= \int_0^\infty \mathrm{e}^{-t}\,\mathrm{E}\,(df(Z) P_t df(Z))\, dt \\
&\le \int_0^\infty \mathrm{e}^{-t}\sqrt{\mathrm{E}\,(df(Z)^2)}\sqrt{\mathrm{E}\,(P_t df(Z)^2)}\, dt \\
&\le \int_0^\infty \mathrm{e}^{-t}\,\mathrm{E}\left(df(Z)^2\right)\, dt = \mathrm{E}\left(df(Z)^2\right)
\end{aligned}$$

□

Proposition 9 (Variance expansion). *If* $Z \sim N(0,1)$ *and* $f \in \mathcal{S}$, *then*

$$\mathrm{Var}\,(f(Z)) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathrm{E}\left(f^{(n)}(Z)\right)^2.$$

If, moreover $\mathrm{E}\left(f^{(n)}(Z)^2\right)/n! \to 0$, *then*

$$\mathrm{Var}\,(f(Z)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}\,\mathrm{E}\left(f^{(n)}(Z)^2\right).$$

Proof. See [NP12-1] pp. 15-16 The Fourier expansion of f is

$$f(Z) - \mathrm{E}\,(f(Z)) = \sum_{n=1}^{\infty} \frac{1}{n!}\,\mathrm{E}\left(f^{(n)}(Z)\right) H_n(Z)$$

and

$$\mathrm{E}\left((f(Z)-\mathrm{E}\,(f(Z)))^2\right) = \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n!}}\,\mathrm{E}\left(f^{(n)}(Z)\right)\right)^2$$

which proves the first part. For the second part, define

$$g(t) = \mathrm{E}\left(Q_{(\sqrt{1-t},\sqrt{t})} f(Z)^2\right), \quad 0 \le t \le 1$$

and note that $\mathrm{Var}\,(f(Z)) = g(1) - g(0)$. Then compute the Taylor expansion. □

Home work

Read all texts below, then chose and solve 2 exercises. The paper is due on Fri May 15.

Exercise 1. Prove Proposition 1 and show that it applies to $f \in \mathcal{S}$

Exercise 2. Prove Proposition 2

Exercise 3. Prove Proposition 3

Exercise 4. Provide a numerical application of Proposition 4

Exercise 5. Prove Proposition 6 items (3-6).

Exercise 6. Prove proposition 7.

Exercise 7. Prove Proposition 9.

As a second option, you can chose among the exercises listed in Sec. 1.7 of the book [NP12-1].

Collegio Carlo Alberto
E-mail address: giovanni.pistone@carloalberto.org