Stocasic Calculus 2012
Part 2

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Change of measure I

Our probability model consist of

- A *sample space* Ω and σ-*algebra* $\mathcal{F}$;
- A *probability measure* $\mathbb{P}$ on the *measurable space* $(\Omega, \mathcal{F})$;
- A *filtration* $\mathcal{F}(t)$, $0 \leq t$, of the *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$;
- A *d-dimensional Browniam motion* $\mathbf{W}(t)$, $t \geq 0$, of the *probability basis* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t \geq 0})$.

Theorem (Probability density)

Let Z be a positive random variable such that $\mathbb{E}(Z) = 1$.

1. $\mathbb{Q}(A) = \mathbb{E}(Z\mathbf{1}_A)$, $A \in \mathcal{F}$, *defines a probability measure on* $(\Omega, \mathcal{F})$.
2. Z *is uniquely detemined by* $\mathbb{P}$ *and* $\mathbb{Q}$ *and is called the density of* $\mathbb{Q}$ *with respect to* $\mathbb{P}$, *written as* $\mathbb{Q} = Z \cdot \mathbb{P}$.
3. *If Z is strictly positive, then* $\mathbb{P} = \frac{1}{Z} \cdot \mathbb{Q}$.
4. $\mathbb{E}_{\mathcal{Q}}[\Phi] = \mathbb{E}_{\mathbb{P}}[Z\Phi]$ *if one of the expectation exists.*

Change of measure II

Example

Let $\mathbb{P} = \mathrm{N}(0, \sigma^2)$ and $\mathbb{Q} = \mathrm{N}(\mu, \sigma^2)$. Then

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\Phi] &= \int \Phi(y) \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{1}{2\sigma^2}(y-\mu)^2} dy \\ &= \int \Phi(y) \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{1}{2\sigma^2}y^2} \mathrm{e}^{\frac{1}{\sigma^2}(\mu y - \frac{1}{2}\mu^2)} dy \\ &= \mathbb{E}_{\mathbb{P}}[Z\Phi], \quad \text{if } Z(y) = \mathrm{e}^{\frac{1}{\sigma^2}(\mu y - \frac{1}{2}\mu^2)}.\end{aligned}$$

Note:

- $\mathbb{Q}$ is the image of $\mathbb{P}$ under the transformation $x \mapsto x + \mu = y$;
- the density Z is strictly positive because is an exponential;
- the exponent of the density has a peculiar affine form;
- Try the bivariate case.

Change of measure III

Theorem (Conditional expectation)

Formula *If* $\mathbb{Q} = Z \cdot \mathbb{P}$, $\Phi \in L^1(\mathbb{Q})$ *and* $\mathcal{G}$ *is a sub-*σ*-algebra, then*

$$\mathbb{E}_{\mathbb{Q}}\left[\Phi|\mathcal{G}\right] = \frac{\mathbb{E}_{\mathbb{P}}\left[Z\Phi|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{P}}\left[Z|\mathcal{G}\right]}.$$

Sufficency *If the density* Z *is* $\mathcal{G}$*-measurable, then*

$$\mathbb{E}_{\mathbb{Q}}\left[\Phi|\mathcal{G}\right] = \mathbb{E}_{\mathbb{P}}\left[\Phi|\mathcal{G}\right].$$

- The formula is a generalization of the conditioning formula for joint densities $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$.
- In the statistical model with likelyhood $f(S(\omega);\theta)$ the conditional expectation with respect to the sufficient syatistics S does not depend on θ.

Change of measure IV

Proof.

- The random variable $\frac{\mathbb{E}_{\mathbb{P}}[Z\Phi|\mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[Z|\mathcal{G}]}$ is well defined and $\mathcal{G}$-measurable.
- If G is bounded and $\mathcal{G}$-measurable,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{\mathbb{E}_{\mathbb{P}}[Z\Phi|\mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[Z|\mathcal{G}]}\right)G\right] &= \mathbb{E}_{\mathbb{P}}\left[Z\left(\frac{\mathbb{E}_{\mathbb{P}}[Z\Phi|\mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[Z|\mathcal{G}]}\right)G\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[Z|\mathcal{G}]\frac{\mathbb{E}_{\mathbb{P}}[Z\Phi|\mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[Z|\mathcal{G}]}G\right] \\
&= \mathbb{E}_{\mathbb{P}}[Z\Phi G] \\
&= \mathbb{E}_{\mathbb{Q}}[\Phi G]
\end{aligned}
$$

□

Martingale measure

Theorem (Martingales under $Z \cdot \mathbb{P}$)

Let $Z(t)$, $0 \leq t \leq T$, be a strictly positive martingale with $Z(0) = 1$. Then $\mathbb{E}_{\mathbb{P}}[Z(T)] = 1$ and we can define $\mathbb{Q}_T = Z(T) \cdot \mathbb{P}$. The adapted process $X(t)$, $0 \leq t \leq T$ is a $\mathbb{Q}_T$-martingale if, and only if, $Z(t)X(t)$, $0 \leq t \leq T$ is a $\mathbb{P}$-martingale.

Proof

$$\mathbb{E}_{\mathbb{Q}_T}[X(t) - X(s)|\mathcal{F}(s)] = \frac{\mathbb{E}_{\mathbb{P}}[Z(T)(X(t) - X(s))|\mathcal{F}(s)]}{\mathbb{E}_{\mathbb{P}}[Z(T)|\mathcal{F}(s)]}$$
$$= \frac{\mathbb{E}_{\mathbb{P}}[Z(t)X(t)|\mathcal{F}(s)] - Z(s)X(s)}{Z(s)}$$

The LHS is zero if, and only if, the RHS's numerator is zero. □

Corollary

If $dX = \Delta d\mathbf{W} + \Theta ds$ and $dZ = \Sigma d\mathbf{W}$, then

$$d(X_t Z_t) = X_t \Sigma_t d\mathbf{W}_t + Z_t \Delta_t d\mathbf{W}_t + Z_t \Theta_t dt + \Delta_t \circ \Sigma_t dt$$

and the condition becomes $Z\Theta + \Delta \circ \Sigma = 0$.

Girsanov's theorem

Theorem

Let W be a Brownian motion and Θ a process such that

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T \Theta^2(u)du\right], \mathbb{E}_{\mathbb{P}}\left[\int_0^T \Theta^2(u)Z^2(u)du\right] < +\infty.$$

Define

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right).$$

Then:

1. $Z(t)$, $0 \leq t \leq T$ *is martingale such that* $\mathbb{E}_{\mathbb{P}}[Z(T)] = 1$ *and* $\mathbb{Q}_T = Z(T) \cdot \mathbb{P}$ *is a probability.*
2. *The process* $\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du$, $0 \leq t \leq T$ *is a* $\mathbb{Q}_T$*-Brownian motion.*

Proof. Use the Corollary and Lévy's theorem. □

Stock under Risk-neutral measure I

- Let the *stock value process* be

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$$

- If $S(0)$ is a constant and $\alpha(t)$ is a deterministic function, then $\mathbb{E}\left(S(t)\right) = S(0)\mathrm{e}^{\int_0^t \alpha(u)du}$, so that $\alpha(t)$ is the *rate of return* of the mean.
- If $S(0)$ is random and $\alpha(t)$ is a process, and $\widetilde{S}(t) = \mathrm{e}^{-\int_0^t \alpha(u)du} S(t)$,

$$\begin{aligned} d\widetilde{S}(t) &= -\alpha(t)\widetilde{S}(t)dt + \alpha(t)\widetilde{S}(t) + \sigma(t)\widetilde{S}(t)dW(t) \\ &= \sigma(t)\widetilde{S}(t)dW(t), \end{aligned}$$

so that $\mathbb{E}\left(\mathrm{e}^{-\int_0^t \alpha(u)du} S(t)\right) = \mathbb{E}\left(S(0)\right)$, then $\alpha(t)$ is the *mean rate of return*.

Stock under Risk-neutral measure II

- By the Ito formula we obtain

$$d(\widetilde{S}(t))^2 = 2\widetilde{S}(t)d\widetilde{S}(t) + \sigma^2(t)\widetilde{S}^2(t)dt,$$

hence $\mathbb{E}\left(\widetilde{S}^2(t)\right) = S^2(0) + \mathbb{E}\left(\int_0^t \sigma^2(u)\widetilde{S}^2(u)du\right)$. The process $\sigma(t)$ is the *volatility*.

- The closed form solution of the Ito equation is

$$S(t) = S(0)\exp\left(\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds\right),$$

hence the process is positive.

- Define the *discount process*

$$D(t) = \mathrm{e}^{-\int_0^t R(s)ds},$$

whose differential is $dD(t) = -R(t)D(t)dt$.

Stock under Risk-neutral measure III

- The *discounted stock price* is

$$D(t)S(t) =$$
$$S(0)\exp\left(\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s)\right) ds\right),$$

whose differential is

$$\begin{aligned} dD(t)S(t) &= (\alpha(t) - R(t)D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\ &= \sigma(t)D(t)S(t)(\Theta(t)dt + dW(t)), \end{aligned}$$

where

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

is the *market price of risk*.

Stock under Risk-neutral measure IV

- Let us apply Girsanov's theorem to the process

$$\widetilde{W}(t) = \int_0^t \Theta(s)ds + W(t).$$

The exponential martingale $Z(t) = \mathrm{e}^{W(t) - \frac{1}{2}\Theta^2(s)ds}$ produces a new probability $\mathbb{Q} = Z(T) \cdot \mathbb{P}$ unde which $\widetilde{(}t)$, $0 \leq\leq T$ is a Brownian motion, and the discounted stock price

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\widetilde{W}(u)$$

is a martingale.

- The undiscounted stock price, as a function of $\widetilde{W}$ is

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t),$$

whose mean return rate is R.