Brownian Motion

1. Conditioning of gaussian vectors

Let $y, x_1, \dots, x_n$ be a gaussian vector. Compute $\hat{a}_1, \dots, \hat{a}_n$ such that

$$y - \sum_{j=1}^{n} \hat{a}_j x_j \text{ is uncorrelated with } x_j,\ j=1\dots n.$$

The condition is

$$\mathrm{Cov}\left(y - \sum_{j=1}^{n} \hat{a}_j x_j, x_i\right) = \mathrm{Cov}(y, x_i) - \sum_{j=1}^{n} \hat{a}_j \mathrm{Cov}(x_j, x_i) = 0, \quad i = 1\dots n.$$

Th.1 $\mathbb{E}(y \mid x_1 \dots x_n) = \mathbb{E}(Y) + \sum_{j=1}^{n} \hat{a}_j (x_j - \mathbb{E}(x_j))$

Proof. $\mathbb{E}(y \mid x_1 \dots x_n) =$

$$\mathbb{E}\left(y - \sum_{j=1}^{n} \hat{a}_j x_j + \sum_{j=1}^{n} \hat{a}_j x_j \mid x_1 \dots x_n\right) =$$

$$\mathbb{E}\left(y - \sum_{j=1}^{n} \hat{a}_j x_j\right) + \sum_{j=1}^{n} \hat{a}_j x_j \qquad \square$$

We here used

1) Uncorrelated gaussian vectors are independent
2) If X an Y are independent, then
$$\mathbb{E}(X \mid Y) = \mathbb{E}(X)$$

Lemma Let X independent of Y $(X \perp\!\!\!\perp Y)$ then

$$\mathbb{E}(\phi(X, Y) \mid X) = \int \phi(X, y) F_Y(dy)$$

Proof. F_Y is the distribution of the random variable Y. Define $\hat{\phi}(x) = \int \phi(x, y) F_Y(dy) = \mathbb{E}(\phi(x, Y))$. The random variable $\hat{\phi}(X) = \int \phi(X, y) F_Y(dy)$ is equal to

$\mathbb{E}(\phi(X,Y)\mid X)$ iff for all function $\psi(X)$

$$\mathbb{E}(\phi(X,Y)\psi(X)) = \mathbb{E}(\hat{\phi}(X)\psi(X))$$

Because of $X \perp\!\!\!\perp Y$, the lhs is

$$\iint \phi(x,y)\psi(x)\, F_X(dx)\, F_Y(dy) =$$

$$\int \left(\int \phi(x,y)\, F_Y(dy) \right) \psi(x)\, F_X(dx) =$$

$$\int \hat{\phi}(x)\psi(x)\, F_X(dx) \qquad \square$$

Th 2 Under the same conditions as the th 1

$$\mathbb{E}(\phi(Y)\mid X_1 \dots X_n) =$$

$$\int \phi\Big(z + \sum_{j=1}^{n} \hat{a}_j X_j\Big)\, F_{Y - \sum_j \hat{a}_j X_j}(dz)$$

Proof.

$$\mathbb{E}(\phi(Y)\mid X_1 \dots X_n) =$$

$$\mathbb{E}\Big(\phi\Big(Y - \sum_{j=1}^{n} \hat{a}_j X_j + \sum_{j=1}^{n} a_j X_j\Big) \,\Big|\, X_1 \dots X_n\Big) =$$

$$\int \phi\Big(z + \sum_{j=1}^{n} a_j X_j\Big)\, F_{Y - \sum_j \hat{a}_j X_j}(dz) \qquad \square$$

Ex 1. Assume $\Gamma_X = [\mathrm{Cov}(X_i, X_j)]_{i,j=1\dots n}$ non-singular. Compute the conditional distribution of Y given $X_1 \dots X_n$

Ex 2. Same with Γ_X singular

2. Brownian motion

D1 (Shreve p. 94) Let $(\Omega, \mathcal{F}, P)$ be a probability space and W a ~~continuous~~ stochastic process with continuous trajectories. W is a BM (or a Wiener process WP) if

1) $W_0 = 0$

2) $0 = t_0 < t_1 < \ldots < t_n$ the increments

$$W_{t_1},\ W_{t_2} - W_{t_1},\ \ldots,\ W_{t_n} - W_{t_{n-1}}$$

are independent ~~a~~ and normally distributed with

$$\begin{cases} \mathbb{E}(W_t - W_s) = 0 \\ \mathbb{V}(W_t - W_s) = t - s \end{cases} \qquad s < t$$

If we define $\mathcal{F}_t = \sigma(W_s, s \le t)$, then the process $(W_t)_{t \ge 0}$ is adapted to $(\mathcal{F}_t)_{t \ge 0}$.

Th (Wiener) $\Omega = C(\mathbb{R}_+)$, $W_t(\omega) = \omega(t)$, there exist $\mathbb{P}$ (the Wiener probability measure) such that W is a BM.

Ex 2. Compute the joint density of $W_{t_1}, \ldots, W_{t_n}$

A more general definition is the following

D2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$ a filtered probability space. A ~~process~~ continuous process W is a BM if for all $s < t$

$W_t - W_s \sim \mathcal{N}(0, t-s)$ and $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$.

Ex 3. Prove the implication i.e. compare D1 with D2.

Properties

1. A BM is a martingale

$$\mathbb{E}(W_t \mid \mathcal{F}_s) = \mathbb{E}(W_t - W_s + W_s \mid \mathcal{F}_s)$$
$$= \mathbb{E}(W_t - W_s) + W_s = W_s$$

2. A BM is a Markov process

$$\mathbb{E}(\varphi(W_t) \mid \mathcal{F}_s) = \mathbb{E}(\varphi(W_t - W_s + W_s) \mid \mathcal{F}_s)$$
$$= \int \varphi(z + W_s) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)} z^2} dz$$

$y = z + W_s$

$$= \int \varphi(y) \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)}(y - W_s)^2} dy$$

that is the conditional distribution of W_t given $\mathcal{F}_s$ is $\mathcal{N}(W_s, t-s)$.

3. Piece-wise linear approximation

A. Let $f: [0,T] \to \mathbb{R}$ be continuous. The n-th piece-wise linear approximation of f is a continuous function $f_n: [0,T] \to \mathbb{R}$ such that

1) $f_n(\frac{k}{N}T) = f(\frac{k}{N}T) \quad k = 0, 1, \dots n$

2) $f_n(t) = f(\frac{k}{N}T) + \left(f(\frac{k+1}{N}T) - f(\frac{k}{N}T)\right)(t - \frac{k}{N}T) \quad t \in [\frac{k}{N}T, \frac{k+1}{n}T]$ (NB: 2) $\to$ 1))

As f is continuous on a bounded interval the modulus of continuity

$$\delta(f, u) = \sup\left\{|f(t) - f(s)| : |t - s| \leq u\right\}$$

is finite and

$$\lim_{u \to 0} \delta(f, u) = 0.$$

We have

$$\sup_t |f(t) - f_n(t)| = \sup_k \sup_{t \in [\frac{k}{m}T, \frac{k+1}{m}T]} |f(t) - f_n(t)|$$

$$= \sup_k \sup_{t \in [\frac{k}{m}T, \frac{k+1}{m}T]} \left| \left(f(t) - f(\tfrac{k}{N}T)\right) + \left(f(\tfrac{k+1}{n}T) - f(\tfrac{k}{n}T)\right)(t - \tfrac{k}{N}T) \right|$$

$$\leq \delta(f, \tfrac{T}{N}) + \delta(f, \tfrac{T}{N})\tfrac{T}{N} = \delta(f, \tfrac{T}{m})(1 + \tfrac{T}{N}) \to 0$$

as $n \to +\infty$.

B. Define $W_t^{(n)}(\omega)$ to be the n-th piece-wise linear approximation of the continuous trajectory $t \mapsto W_t(\omega)$, $t \in [0,T]$. Then

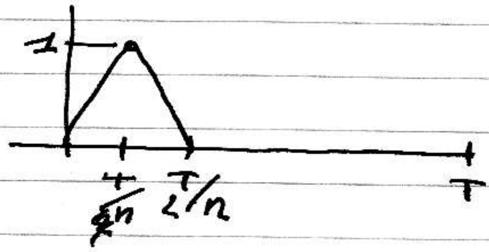
$\sup_{t \in [0,T]} | W_t(\omega) - W_t^{(n)} | = 0$ for all

ω such that $t \mapsto W_t(\omega)$ is continuous.

Note that $W_t^{(n)}$ is a gaussian process.

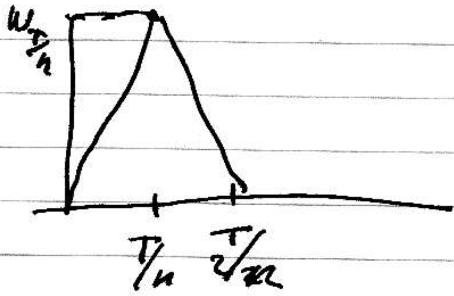
c. Consider the function ("wavelet")

$H_n(t) =$



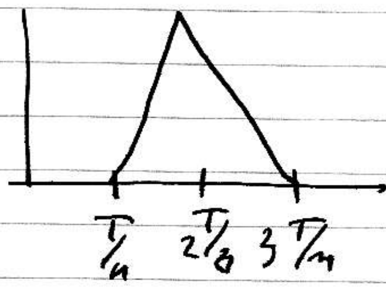
then

$W_{T/n} \, H_n(t) =$



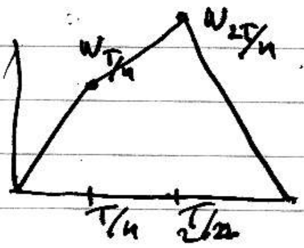
We have

$H_n(t - \frac{T}{n}) =$



and

$W_{T/n} H_n(t) + W_{\frac{2T}{n}} H_n(t - \frac{T}{n}) + \dots$



(...)

In general

$$W_t^{(n)} = \sum_{k=1}^{n} W_{\frac{k}{n}T}\, H_n\left(t - \frac{k-1}{n}T\right) \qquad t \in [0,T]$$

In term of the increments, we have

$$W_{\frac{k}{n}T} - W_{\frac{k-1}{n}T} = \sqrt{\frac{T}{n}}\, Z_k \quad \text{with}$$

Z_k $k=1,\dots,n$ IID $N(0,1)$. By adding up all increments,

$$W_{\frac{k}{n}T} = \sqrt{\frac{T}{n}}\left(Z_1 + \dots + Z_k\right)$$

$$W_t^{(n)} = \sum_{k=1}^{n} \sqrt{\frac{T}{n}} \sum_{h=1}^{k} Z_h\, H_n\left(t - \frac{k-1}{n}T\right)$$

$$\sqrt{\frac{T}{n}} \sum_{h=1}^{n} Z_h \underbrace{\sum_{k=h}^{n} H_n\left(t - \frac{k-1}{n}T\right)}_{\text{Ex 4 ?}}$$

Ex5. Compute $Cov(W_s^{(n)}, W_t^{(n)})$, $s<t$

D. Recursive $\boxed{\text{simulation}}$ of $W^{(1)}, W^{(2)}, \dots, W^{(n)}, \dots$

Assume $T=1$. The sequence $Z_1, Z_2, \dots$ is IID $N(0,1)$ and (potentially) infinite.

Then $Z_1 \sim W_1$ and

$$W_t^{(1)} \sim t Z_1$$

4. Quadratic variation

Consider a partition of $[0,t]$, $t_k = \frac{k}{n} t$, $k = 0, 1, \dots, n$.

$$\sum_{k=1}^{n} \left(W_{t_k} - W_{t_{k-1}} \right)^2 = Q_t^{(n)}. \quad \text{Each}$$

$$W_{t_k} - W_{t_{k-1}} \sim N\left(0, \frac{t}{n}\right), \text{ i.e.}$$

$$W_{t_k} - W_{t_{k-1}} = \sqrt{\frac{t}{n}}\, Z_k$$

$$Z_1 \dots Z_n \quad \text{IID} \quad N(0,1)$$

$$Q_t^{(n)} = \sum_{k=1}^{n} \left(\sqrt{\frac{t}{n}}\, Z_k \right)^2 = \frac{t}{n} \sum_{k=1}^{n} Z_k^2$$

By a <u>version</u> of the Law of Large numbers

$$Q_t^{(n)} \to t\, \mathbb{E}(Z_1^2) = t$$

Th. 1) W_t^2 is a sub-martingale, i.e

$$\mathbb{E}(W_t^2 \mid \mathcal{F}_s) \geq W_s^2$$

2) $W_t^2 - t$ is a martingale

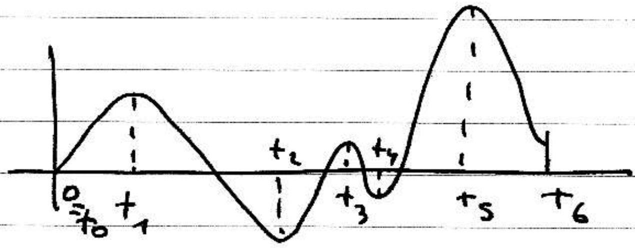
3) $e^{a W_t - \frac{a^2}{2} t}$ is a Martingale

Proof Ex. 7.

Variation of a function

the variation of the trajectory

$x(t) =$



is

$$|x(t_1)-x(t_0)| + |x(t_2)-x(t_1)| + |x(t_3)-x(t_2)|$$
$$+ |x(t_4)-x(t_3)| + |x(t_5)-x(t_4)| + |x(t_6)-x(t_5)|$$

More generally, we check the existence of the limit along the partitions

$$\pi = t_0=0, t_1, t_2, \dots, t_n=t$$

of the total sum of absolute increments

$$\lim_{\pi} \sum_{i=1}^{n} |x(t_i)-x(t_{i-1})|$$

Assume $x \in C^1$. By Lagrange (!) MVT

$t_{i-1}\ s_i\ t_i$

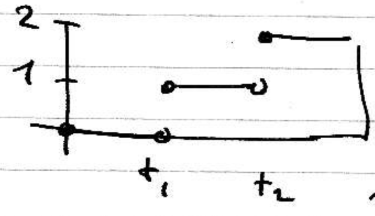
$$x(t_i) - x(t_{i-1}) = x'(s_i)(t_i - t_{i-1})$$

and

$$\sum_{i=1}^{n} |x'(s_i)|(t_i - t_{i-1}) \text{ is convergent}$$

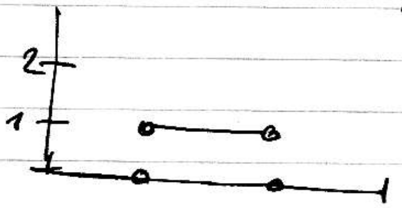
to $\int_0^t |x'(s)|\,ds$.

Finite variation does not require continuity. A step function (trajectory) has finite variation, e.g.



has variation $= 2$, as it is nondecreasing, and e.g.

has variation 2.



This is of interest for processes with <u>jumps</u>.

It is more convenient to free the running time t from the partition, i.e. we take

$$\pi = 0 < t_1 < t_2 < \ldots < t_n < \ldots \qquad t_n \to +\infty$$

and compute the variation at each time t as

$$\lim_{\pi} \sum_i{}' \left| x(t \wedge t_i) - x(t \wedge t_{i-1}) \right|$$

as

for	$t \leq t_{i-1}$	we have	$x(t) - x(t) = 0$
"	$t_{i-1} < t < t_i$	"	$x(t) - x(t_{i-1})$
"	$t_i \leq t$	"	$x(t_i) - x(t_{i-1})$

<u>Basic computation</u>: Assume x finite variation and continuous. Fix a partition. By telescoping and Taylor formula:

$$x(t)^2 - x(0)^2 = \sum_{i=1}^{\infty} x^2(t\wedge t_i) - x^2(t\wedge t_{i-1})$$

$$= \sum_{i=1}^{\infty} 2\, x(t\wedge t_{i-1})(x(t\wedge t_i) - x(t\wedge t_{i-1}))$$

$$+ \sum_{i=1}^{\infty} |x(t\wedge t_i) - x(t\wedge t_{i-1})|^2.$$

The last term is bounded by

$$\sup_i |x(t\wedge t_i) - x(t\wedge t_{i-1})| \times \sum_{i=1}^{\infty} |x(t\wedge t_i) - x(t\wedge t_{i-1})|$$

where the first factor goes to zero (continuity) and the second factor is finite (finite variation)

It follows the existence of the limit

$$\lim_{\pi} 2 \sum_{i=1}^{\infty} x(t\wedge t_{i-1})(x(t\wedge t_i) - x(t\wedge t_{i-1}))$$

$$\equiv 2\int_0^t x(t)\, dx(t)$$

and the formula

$$x^2(t) - x(0)^2 = 2\int_0^t x(t)\, dx(t)$$

Ex 3. Compute $\varphi(x(t))$ where $\varphi \in C^2$ with bounded derivatives

Quadratic Variation

7/

W is a Wiener Process (Brownian motion) on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. Fix a partition

$$0 = t_0 < t_1 < \dots < t_n = t$$

and compute (telescoping and Taylor expansion)

$$W_t^2 - W_0^2 = \sum_{i=1}^{n} W_{t_{i+1}}^2 - W_{t_i}^2$$

$$= 2\sum_{i=1}^{n} W_{t_i}(W_{t_{i+1}} - W_{t_i}) + \sum_{i=1}^{n} (W_{t_{i+1}} - W_{t_i})^2$$

Both LHS and RHS are stochastic processes. In order to see the dependence on t in the RHS we change the notation as follows

$$0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} \dots$$

with $t_n \to \infty$ as $n \to +\infty$ and write

$$W_t^2 - W_0^2 = \sum_i W_{t\wedge t_{i+1}}^2 - W_{t\wedge t_i}^2$$

$$= \sum_i 2\, W_{t\wedge t_i}(W_{t\wedge t_{i+1}} - W_{t\wedge t_i}) + \sum_i (W_{t\wedge t_{i+1}} - W_{t\wedge t_i})^2$$

To check the formulae consider $t \in]t_i, t_{i+1}]$.

The continuous process

$$M_t^\pi = \sum_{i} 2\, W_{t\wedge t_i}(W_{t\wedge t_{i+1}} - W_{t\wedge t_i})$$

is a MARTINGALE. In fact consider one term of the sum and check the conditioning in each case:

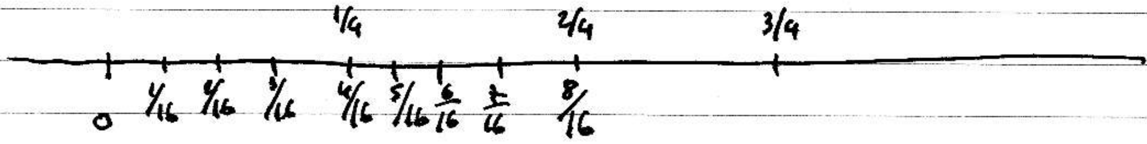
$$2W_{t\wedge t_i}\left(W_{t\wedge t_{i+1}} - W_{t\wedge t_i}\right) = \begin{cases} 0 & t\le t_i \\ 2W_{t_i}\left(W_t - W_{t_i}\right) & t_i<t\le t_{i+1} \\ 2W_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) & t_{i+1}<t \end{cases}$$

Let us compare two approximations, one of step 2^{-m} and the other with step 2^{-n}, $m<n$.

$$M_t^{(m)} = \sum_k 2W_{t\wedge\frac{k}{2^m}}\left(W_{t\wedge\frac{k+1}{2^m}} - W_{t\wedge\frac{k}{2^m}}\right)$$

$$M_t^{(n)} = \sum_k 2W_{t\wedge\frac{k}{2^m}}\left(W_{t\wedge\frac{k+1}{2^n}} - W_{t\wedge\frac{k}{2^n}}\right)$$

$$M_t^{(n)} - M_t^{(m)} = \sum_{k=1}^{\infty} 2\left(W_{t\wedge\frac{k}{2^n}} - W_{t\wedge\frac{[k]}{2^n}}\right)\left(W_{t\wedge\frac{k+1}{2^n}} - W_{t\wedge\frac{k}{2^n}}\right)$$



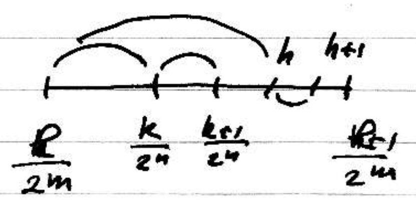
Let us compute $\mathbb{E}\left(M_t^{(n)} - M_t^{(m)}\right)^2 =$

$$\sum_{h,k=1}^{\infty} 4\,\mathbb{E}\Big[\left(W_{t\wedge\frac{k}{2^n}} - W_{t\wedge\frac{[k]}{2^n}}\right)\left(W_{t\wedge\frac{k+1}{2^n}} - W_{t\wedge\frac{k}{2^n}}\right) \times \left(W_{t\wedge\frac{h}{2^n}} - W_{t\wedge\frac{[h]}{2^n}}\right)\left(W_{t\wedge\frac{h+1}{2^n}} - W_{t\wedge\frac{h}{2^n}}\right)\Big]$$

Case ~~...~~ $h=k$

$$= \sum_{k=1}^{\infty} 4\, \mathbb{E}\left[\left(W_{t\wedge \frac{k}{2^n}} - W_{t\wedge \frac{(k)}{2^n}}\right)^2 \left(W_{t\wedge \frac{k+1}{2^n}} - W_{t\wedge\frac{k}{2^n}}\right)^2\right]$$

Case $h \neq k$:



$$= 4 \sum_{k=1}^{\infty} \mathbb{E}\left(W_{t\wedge \frac{k}{2^n}} - W_{t\wedge \frac{(k)}{2^n}}\right) \left(t\wedge \tfrac{k+1}{2^n} - t\wedge\tfrac{k}{2^n}\right)$$
$$\left(t\wedge \tfrac{k}{2^n} - t\wedge \tfrac{(k)}{2^n}\right)$$

$\leq 4t\, 2^{-m}$ Ex 4. Check the proof in Shreve p.103

Conclusion

1. The martingale $M^{(n)} - M^{(m)}$ is such that
$\mathbb{E}\left(\left(M_t^{(n)} - M_t^{(m)}\right)^2\right) \to 0$ as $m,n \to +\infty$
2. The Doob maximal lemma implies
$$\mathbb{E}\left(\sup_{s\leq t} |M_s^{(n)} - M_s^{(m)}|^2\right) \to 0 \quad \text{as } m,n \to \infty$$
3. The limit $M^{(n)} \to 2\int^t W_s\, dW_s$ is a continuous, adapted MARTINGALE.
4. The processes $\sum_i \left(W_{t\wedge t_i} - W_{t\wedge t_{i-1}}\right)^2$ converge in the sense 2. to the quadratic variation t which is a compensator of W_t^2.