Probability 2020
Recap of Measure Theory

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Draft February 9, 2020

Some books

There will be written notes of this course. Howevere, it is useful to have a set of standard textbook to which refer in case of need..

- P. Malliavin. *Integration and probability*, volume 157 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995. With the collaboration of Héléne Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky is a textbook that presents topics in integration theory and calculus. In particular it contains specific probability topics such as, conditioning, martingales, Gaussian spaces.
- S. M. Ross. *Introduction to probability models*. Academic Press, London, 12th edition, 2019 is a very popular introductory textbook oriented to applications and with many examples of contemporary applications.
- W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987 is a classical textbook that presents integration theory and othe topics in calculus that are of interest in probability.

Measurable space

Definition

- A family $\mathcal{B}$ of subsets of S is an field (or a Borel algebrab) on S if it contains $\emptyset$ and S, and it is stable for the complements, finite unions, and finite intersection.
- A family $\mathcal{F}$ of subsets of S is a σ-field on S if it is an field on S and it is stable for denumerable unions and intersections.
- A measurable space is a couple $(S, \mathcal{F})$, where S is a set and $\mathcal{F}$ is a σ-field on S.

- Given the family $\mathcal{C}$ of subsets of S, the σ-field generated by $\mathcal{C}$ is $\sigma(\mathcal{C}) = \cap \{\mathcal{A} \,|\, \mathcal{C} \subset \mathcal{A}$ and $\mathcal{A}$ is a σ-field$\}$.
- Examples: the field generated by a finite partition; the Borel σ-field of $\mathbb{R}$ is generated by the open intervals, or by the closed intervals, or by the intervals, or by the open sets, or by semi-infinite intervals. The Borel σ-field of a metric space is generated by the open sets.

§1.1-2 of Malliavin..

Measure space

Definition

- A measure μ of the measurable space $(S, \mathcal{F})$ is a mapping $\mu \colon \mathcal{F} \to [0, +\infty]$ such that $\mu(\emptyset) = 0$ and for each sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint elements of $\mathcal{F}$, $\mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{i=1}^{\infty} \mu(A_n)$.
- A measure is finite if $\mu(S) < +\infty$; a measure is σ-finite if there is a sequence $(S_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\cup_{n\in\mathbb{N}} S_n = S$ and $\mu(S_n) < +\infty$ for all $n \in \mathbb{N}$.
- A probability measure is a finite measure such that $\mu(S) = 1$; a probability space is the triple $(S, \mathcal{F}, \mu)$, where μ is a probability measure.

- Examples: Lebesgue measure; probability measure on a partition; probability measure on a denumerable set; Lebesgue measure on the unit cube.
- Equivalently, a probability measure is finitely additive and sequentially continuous at $\emptyset$.

§1.3 of Malliavin.

Monotone classes I

- A seguence of sets $(A_n)_{n\in\mathbb{N}}$ is monotone if $A_n \subset A_{n+1}$, or $A_n \supset A_{n+1}$. In such a case, $\lim_{n\to\infty} A_n = \cup_n A_n$, respectively $\lim_{n\to\infty} A_n = \cap_n A_n$.
- A class of sets $\mathcal{M}$ in monotone is for all monotone sequence (A_n) in $\mathcal{M}$ the limit is in $\mathcal{M}$, $\lim_{n\to\infty} A_n \in \mathcal{M}$.
- A σ-algebra is a monotone set. The intersection of monotone classes is a monotone class.
- Given a class of sets $\mathcal{C}$, the intersection of all monotone classes that contain $\mathcal{C}$ is the monotone class generated by $\mathcal{C}$, $m(\mathcal{C})$.

Theorem (of monotone classes)

The σ algebra generated by a field $\mathcal{B}$ is equal to the monotone class generated by $\mathcal{B}$, $\sigma(\mathcal{B}) = m(\mathcal{B})$

See Malliavin § 1.4 and a more sophisticated version below. This result is used in almost every proof of uniqueness for measures. It is also relevant the corrisponding result for classes of real random variables.

Monotone classes II

Proof.

1. $m(\mathcal{B}) \subset \sigma(\mathcal{B})$ is trivial.
2. Define $\Phi(A) = \{B \mid A \cup B, A \setminus B, B \setminus A \in m(\mathcal{B})\}$ and check that $B \in \Phi(A)$ is equivalent to $A \in \Phi(B)$.
3. Check that $\Phi(A)$ is a monotone class and contains $m(\mathcal{B})$.
4. Check that $m(\mathcal{B})$ is a σ-algebra.

□

Theorem (of uniqueness)

If two finite measures are equal on a field $\mathcal{B}$, they are equal on $\sigma(\mathcal{B})$.

Proof.

Show that the class of events where the measures are equal is a monotone class that contains a field. □

Product system aka π-system

Definition

Let S be a set. A π-system on S is a family $\mathcal{I}$ of subsets of S which is stable under finite intersection.

- Examples: the family of all points of a finite set and the empty set; the family of open intervals of $\mathbb{R}$; the famililly of closed intervals of $\mathbb{R}$; the family of cadlàg intervals of $\mathbb{R}$; the family of convex (resp. open convex, closed convex) subsets of $\mathbb{R}^2$; the family of open (resp. closed) set in a topological space.
- If $\mathcal{I}_i$ is a π-system of S_i, $i = 1, \dots, n$, then $\{\times_{i=1}^n I_i \mid I_i \in \mathcal{I}_i\}$ is a π-system of $\times_{i=1}^n S_i$.
- The family of all real functions of the form $\alpha_0 + \sum_{j=1}^n \alpha_j \mathbf{1}_{I_i}$, $n \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, $j = 0, \dots, n$ is a vector space and it is stable for multiplication.

Dynkin system aka d-system

Definition

Let S be a set. A *d-system* on S is a family $\mathcal{D}$ of subsets of S such that

1. $S \in \mathcal{D}$
2. If $A, B \in \mathcal{D}$ and $A \subset B$, then $B \setminus A \in \mathcal{D}$. (Notice that $S \setminus A = A^c$)
3. If $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence in $\mathcal{D}$, then $\cup_{n \in \mathbb{N}} \in \mathcal{D}$

- Given probabilities μ_i and $i = 1, 2$ on the measurable space $(S, \mathcal{F})$, the family $\mathcal{D} = \{A \in \mathcal{F} \mid \mu_1(A) = \mu_2(A)\}$ in a d-system.
- Given measurable spaces $(S_i, \mathcal{F}_i)$, $i = 1, 2$, the product space $(S, \mathcal{F}) = (S_1 \times S_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma \{A_1 \times A_2 \mid A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$, and $x \in S_1$, the family $\mathcal{D} = \{A \in \mathcal{F}_1 \otimes \mathcal{F}_2 \mid A \cap \{x\} \times S_2 = \{x\} \times A_x, A_x \in \mathcal{F}_2\}$ is a d-system.

Dynkin's lemma

Theorem

1. *A family of subsets of S is a σ-field if, and only if, it is both a d-system and a π-system.*
2. *If $\mathcal{I}$ is a π-system, then $d(\mathcal{I}) = \sigma(\mathcal{I})$.*
3. *Any d-system that contains a π-system contains the σ-field generated by the π-system.*

Theorem

If two probability measures on the same measurable space agree on a π-system $\mathcal{I}$ they are equal on $\sigma(\mathcal{I})$.

Probability space

Definition

A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ of a sample space Ω (set of possible worlds), a σ-field $\mathcal{F}$ on Ω, a probability measure $\mathbb{P}\colon \mathcal{F} \to [0,1]$. An element $\omega \in \Omega$ is a sample point (world); an element $A \in \mathcal{F}$ is an event; the value $\mathbb{P}(A)$ is the probability of the event A.

- Examples: a finite set, all its subsets, a probability function $p\colon \Omega \to \mathbb{R}_{>0}$ such that $\sum_{\omega\in\Omega} p(\omega) = 1$; $\mathbb{Z}_{\geq}$ with all its subsets, and a probability function $p\colon \mathbb{Z}_{\geq} \to \mathbb{R}_{>0}$ such that $\sum_{k=0}^{\infty} p(k) = 1$; the restriction of a probability space to a sub-σ-field; the product of two probability spaces.
- Bernoulli trials. Let $\Omega = \{0,1\}^{\mathbb{N}}$ and let $\mathcal{F}_n = \left\{ A \times \{0,1\} \times \{0,1\} \times \cdots \,\middle|\, A \subset \{0,1\}^n \right\}$, $\mathcal{F} = \sigma(\mathcal{F}_n \colon n \in \mathbb{N})$. Given $\theta \in [0,1]$, the function $p_n(x_1 x_2 \cdots x_n \cdots) = \theta^{\sum_{i=1}^n x_i}(1-\theta)^{n-\sum_{i=1}^n x_i}$ uniquely defines probability spaces $(\Omega, \mathcal{F}_n, \mathbb{P}_n)$, $n \in \mathbb{N}$, such that $\mathbb{P}_{n+1}|_{\mathcal{F}_n} = \mathbb{P}_n$, hence a probability measure $\mathbb{P}$ on $\mathcal{F}$.

$\limsup$ and $\liminf$

Definition

- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

$$\limsup_{n\to\infty} a_n = \wedge_{m\in\mathbb{N}} \vee_{n\geq m} a_n \quad \text{(maximum limit)}$$

$$\liminf_{n\to\infty} a_n = \vee_{m\in\mathbb{N}} \wedge_{n\geq m} a_n \quad \text{(minimum limit)}$$

- Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of events in the measurable space $(\Omega, \mathcal{F})$.

$$\limsup_{n\to\infty} E_n = \cap_{m\in\mathbb{N}} \cup_{n\geq m} E_n \quad (E_n \text{ infinitely often})$$

$$\liminf_{n\to\infty} E_n = \cup_{m\in\mathbb{N}} \cap_{n\geq m} E_n \quad (E_n \text{ eventually})$$

A similar definition applies to sequences of functions. If $(f_n)_n$ is a sequence of non-negative functions, then the set of $x \in S$ such that $\lim_n f_n(x) = 0$ is equal to the set $\{\limsup_n f_n = 0\}$.

Fatou lemma

Theorem

$$\mathbb{P}\left(\liminf_{n\to\infty} E_n\right) \le \liminf_{n\to\infty} \mathbb{P}(E_n) \le \limsup_{n\to\infty} \mathbb{P}(E_n) \le \mathbb{P}\left(\limsup_{n\to\infty} E_n\right)$$

- $(\limsup_n E_n)^c = \liminf_n E_n^c$; $\limsup_n \mathbf{1}_{E_n} = \mathbf{1}_{\limsup_n E_n}$.
- Proof of FL. Write $\cup_m \cap_{n\ge m} E_n = \cup_m G_m$ so that $G_m \uparrow G = \liminf_n E_n$. We have $\mathbb{P}(G_m) \le \wedge_{n\ge m} \mathbb{P}(E_n)$; monotone continuity (increasing) implies $\mathbb{P}(G_m) \uparrow \mathbb{P}(G)$ hence, $\vee_m \mathbb{P}(G_m) = \mathbb{P}(G)$. The middle inequality is a property of lim inf and lim sup. The least inequality follows from a similar proof using continuity on decreasing sequences or, by taking the complements.
- BC1. Assume $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < +\infty$. We have for all $m \in \mathbb{N}$ that

$$\mathbb{P}\left(\limsup_n E_n\right) \le \mathbb{P}(\cup_{n\ge m} E_n) \le \sum_{n=m}^{\infty} \mathbb{P}(E_n) \to 0 \quad \text{if } m \to \infty$$

hence $\mathbb{P}(\limsup_n E_n) = 0$.

Measurable function

Definition

Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 1, 2$, we say that the function $h\colon S_1 \to S_2$ is measurable, or is a random variable, if for all $B \in \mathcal{S}_2$ the set $h^{-1}(B) = \{s \in S_1 \mid h(s) \in B\}$ belongs into $\mathcal{S}_1$.

Theorem

- *Let $\mathcal{C} \subset \mathcal{S}_2$ and $\sigma(C) = \mathcal{S}_2$. If $h^{-1}\colon \mathcal{C} \to \mathcal{S}_1$, then h is measurable.*
- *Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 1, 2, 3$, if both $h\colon S_1 \to S_2$, $g\colon S_2 \to S_3$ are measurable functions, then $g \circ f : S_1 \to S_3$ is a measurable function.*
- *Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 0, 1, 2$ and $h_i\colon S_0 \to S_j$, $j = 1, 2$, consider $h = (h_1, h_2)\colon S_0 \to S_1 \times S_2$. with product space $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$, Then both h_1 and h_2 are measurable if, and only if, h is measurable.*

Image measure

Definition

Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 1, 2$, a measurable function $h\colon S_1 \to S_2$, and a measure μ_1 on $(S_1, \mathcal{S}_1)$, then $\mu_2 = \mu_1 \circ h^{-1}$ is a measure on $(S_2, \mathcal{S}_2)$. We write $h_{\#}\mu_1 = \mu_2 \circ h^{-1}$ and call it image measure. If μ_1 is a probability measure, we say that $h_{\#}\mu_1$ is the distribution of the random variable h.

- Bernoulli scheme Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the Bernoulli scheme, and define $X_t\colon \Omega \to \{0, 1\}$ to be the t-projection, $X_t(x_1 x_2 \cdots) = x_t$. It is a random variable with Bernoulli distribution $\mathrm{B}(\theta)$. The random variable $Y_n = X_1 + \cdots + X_n$ has distribution $\mathrm{Bin}(\theta, n)$. The random variable $T = \inf\{k \in \mathbb{N} \mid X_k = 1\}$ has distribution $\mathrm{Geo}(\theta)$.

Real random variable

Definition

Let $(S, \mathcal{S})$ be a measurable space. A real random variable is a real function $h\colon S \to \mathbb{R}$ with is measurable into $(\mathbb{R}, \mathcal{B})$.

Theorem

- *$h\colon S \to \mathbb{R}$ is a real random variable if, and only if, for all $c \in \mathbb{R}$ the level set $\{s \in S\}\, h(s) \leq c$ is measurable. The same property holds with $\leq$ replaced by $<$ or $\geq$ or $>$. The condition can be taken as a defintion of extended random variable i.e.* $h\colon S \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
- *If $g, h\colon S \to \mathbb{R}$ are real random variables and $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $\Phi \circ (g, h)$ is a real random variable.*
- *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of real random variables on $(S, \mathcal{S})$. Then* $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ *are real random variable.*

A monotone-class theorems

Theorem

Let $\mathcal{H}$ be a vector space of bounded real functions of a set S and assume $\mathbf{1} \in \mathcal{H}$. Assume

1. *$\mathcal{H}$ is a monotone class i.e., if for each bounded increasing sequence $(f_n)_n \in \mathbb{N}$ in $\mathcal{H}$ the function $\vee_n f_n$ belong to $\mathcal{H}$.*
2. *$\mathcal{H}$ contains the indicator functions of a π-system $\mathcal{I}$.*

Then, $\mathcal{H}$ contains all bounded measurable functions of $(S, \sigma(I))$.

- Application. Consider measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$. Define $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{I} = \{A_1 \times A_2 \,|\, A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{I})$. Let $\mathcal{H}$ be the set of all bounded real funtions $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ such that for each fixed $x \in \Omega_1$ the mapping $\Omega_2 \ni y \mapsto f(x, y)$ is $\mathcal{F}_2$-measurable and for each fixed $y \in \Omega_2$ the mapping $\Omega_1 \ni x \mapsto f(x, y)$ is $\mathcal{F}_1$-measurable.

Simple functions

Let $(S, \mathcal{F})$ be a measurable space.

Definition

A measurable real function, $f \colon S \to \mathbb{R}$, $f^{-1} \colon \mathcal{B} \to \mathcal{F}$, is simple if it takes a finite number of values; equivalently, it is of the form $f = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k}$, $a_k \in \mathbb{R}$, $A_k \in \mathcal{F}$, $k = 1, \dots, m$, $m \in \mathbb{N}$. The algebra with unity of all simple functions is denoted by $\mathcal{S}$; the cone of all non-negative simple function is denoted by $\mathcal{S}_+$.

- Both $\mathcal{S}$ and $\mathcal{S}_+$ are closed for $\vee$ and $\wedge$; $f = f^+ - f^-$, $f \in \mathcal{S}$.
- If f is measurable and non-negative, there exist an increasing sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{S}_+$ such that $\lim_{n \to \infty} f_n(s) = f(s)$, $s \in S$.
- If f is measurable and bounded, there exist an sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{S}$ such that $\lim_{n \to \infty} f_n = f$ uniformely.

Integral of a non-negative function

Let $(S, \mathcal{F}, \mu)$ be a measure space.

Definition

- If $f \in \mathcal{S}$, $f = \sum_{k=1}^{m} a_k \mathbf{1}_{A_k}$, we define its integral to be

$$\int f \ d\mu = \sum_{k=1}^{m} a_k \mu(A_k) \quad \text{where } 0 \cdot \infty = \infty \cdot 0 = 0$$

- If $f : S \to [0, +\infty]$ is measurable, namely $f \in \mathcal{L}_+$, we define its integral to be

$$\int f \ d\mu = \sup \left\{ \int h \ d\mu \,\middle|\, h \in \mathcal{S}_+, h \leq f \right\}$$

- The integral is linear and monotone on $\mathcal{S}^1 = \left\{ f \in \mathcal{S} \,\middle|\, \int f^+ \ d\mu, \int f^- \ d\mu \leq \infty \right\}$. The integral is convex and monotone on $\mathcal{L}_+$.
- If $f \in \mathcal{L}_+$ and $\int f \ d\mu = 0$, then $\mu \{f > 0\} = 0$.

Monotone-Convergence Theorem

Theorem (MON)

let $(f_n)_{n\in\mathbb{N}}$ *be a non-decreasing sequence in* $\mathcal{L}_+$*. Then the pointwise limit* $f = \lim_{n\to\infty} f_n$ *belongs to* $\mathcal{L}_+$ *and* $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

- A sequence of simple functions converging to f is always available.
- If $\alpha, \beta \in \mathbb{R}_{>0}$, $f, g \in \mathcal{L}_+$, then

$$\int (\alpha f + \beta g) \, d\mu = \int \alpha f \, d\mu + \int \beta g \, d\mu$$

- Exercise: If $\mu(S) = 1$, then $\int f \, d\mu = \int_0^\infty \mu\{f > u\} \, du$.

Fatou Lemmas

Theorem (FATOU)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}_+$.

1. $\int (\liminf_{n\to\infty} f_n)\ d\mu \le \liminf_{n\to\infty} \int f_n\ d\mu.$
2. *If, moreover,* $f_n \le g$, $n \in \mathbb{N}$, *and* $\int g\ d\mu < \infty$, *then* $\int (\limsup_{n\to\infty} f_n)\ d\mu \ge \limsup_{n\to\infty} \int f_n\ d\mu.$

- Exercise: Prove 1. by observing that $\limsup$ is the limit of an increasing sequence.
- Exercise: If $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{L}_+$ and $\int f_1\ d\mu < \infty$, then $\int (\lim_{n\to\infty} f_n)\ d\mu = \lim_{n\to\infty} \int f_n\ d\mu$.
- Exercise: If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}_+$ and $f_n \le g$, $n \in \mathbb{N}$, $\int g\ d\mu < \infty$, then $\int (\lim_{n\to\infty} f_n)\ d\mu = \lim_{n\to\infty} \int f_n\ d\mu$.

Integrability

Definition

- Let $\mathcal{L}^1$ be the vector space of measurable real functions such that

$$\int |f| \ d\mu = \int f^+ \ d\mu + \int f^- \ d\mu < \infty$$

- Define the integral to be the linear mapping

$$\mathcal{L}^1 \ni f \mapsto \int f \ d\mu = \int f^+ \ d\mu - \int f^- \ d\mu \in \mathbb{R}$$

- Exercise. Revise L^1-convergence and Dominated Convergence Theorem.

Expectation

- Let $\mathrm{E}\colon \mathcal{L}^{\infty}(S,\mathcal{S}) \to \mathbb{R}$ be such that
  - $\mathbb{E}(\mathbf{1}) = 1$.
  - E is linear and positive (hence monotone).
  - E is continuous on non-increasing sequence converging to 0.
- Every such E defines a probability measure when restricted to indicators, $\mathbb{P}(A) = \mathbb{E}(\mathbf{1}_A)$ and $\mathbb{E}(f) = \int f \; d\,\mathbb{P}$
- A similar observation holds for a $\mathrm{E}\colon \mathcal{L}_{+}(S,\mathcal{S})$.
- If $f \in \mathcal{L}(S,\mathcal{L})$, as $f = f_{+} - f_{-}$ and $|f| = f_{+} + f_{-}$, if $\mathbb{E}(|f|) < \infty$ then $\mathbb{E}(f_{+}), \mathbb{E}(f_{-}) < \infty$. In such a case, we say that $f \in \mathcal{L}^{1}(S,\mathcal{S},\mathbb{P})$ and define $\mathbb{E}(f) = \mathbb{E}(f_{+}) - \mathbb{E}(f_{-})$.
- $\mathrm{E}\colon \mathcal{L}^{1}(S,\mathcal{S},\mathbb{P}) \to \mathbb{R}$ is positive, linear, normalized, continuous for the bounded pointwise convergence.
- Exercise: carefully check everything!

Densities

Let be given a measure space $(S, \mathcal{F}, \mu)$ and a measurable non-negative mapping $p\colon S \to \mathbb{R}$ such that $\int p \, d\mu < \infty$.
The set function

$$p \cdot \mu\colon \mathcal{F} \to \mathbb{R}_{>0}, \quad A \mapsto \int \mathbf{1}_A p \, d\mu$$

is a bounded measure. In fact: $p \cdot \mu(\emptyset) = \int 0 \, d\mu = 0$; given a sequence $(A_n)_{n \in \mathbb{N}}$ of disjoint events, then MON implies

$$p \cdot \mu(\cup_{n \in \mathbb{N}} A_n) = \int \mathbf{1}_{\cup_{n \in \mathbb{N}} A_n} p \, d\mu = \int \sum_{n \in \mathbb{N}} \mathbf{1}_{A_n} \, d\mu = \sum_{n \in \mathbb{N}} \int \mathbf{1}_{A_n} \, d\mu = \sum_{n \in \mathbb{N}} p \cdot \mu(A_n)$$

Exercise: If $f\colon S \to \mathbb{R}$ is measurable and $fp \in \mathcal{L}^1(S, \mathcal{F}, \mu)$, then $f \in \mathcal{L}^1(S, \mathcal{F}, p \cdot \mu)$ and $\int f \, d(p \cdot \mu) = \int fp \, d\mu$. [Hint: try first simple functions, then use MON]

Inequalities I

- Expectation is a positive operator, hence it preserves the order. Most common application is a family of inequalities whose simplest form is Markov inequality: If $x \geq 0$ and $a > 0$, then $\mathbf{1}_{[a,+\infty[} \leq a^{-1}x$. It follows that for each non-negative random variable X we have $\mathbb{P}(X \geq a) \leq a^{-1}\,\mathbb{E}(X)$.

- The previous inequality can be optimised to get, for example, the exponential Markov inequality. Observe that for all $t > 0$ it holds $\{X \geq a\} = \left\{\mathrm{e}^{tX} \geq \mathrm{e}^{ta}\right\}$. It follows that

$$\mathbb{P}(X \geq a) \leq \mathrm{e}^{-ta}\,\mathbb{E}\left(\mathrm{e}^{tX}\right) = \exp\left(-\left(ta - \log \mathbb{E}\left(\mathrm{e}^{tX}\right)\right)\right).$$

  If $I(a) = \sup_{t>0}\left(ta - \log \mathbb{E}\left(\mathrm{e}^{tX}\right)\right)$, then $\log \mathbb{P}(X \geq a) \leq -I(a)$.

- Jensen inequality: Let $\Phi \colon \mathbb{R} \to \overline{\mathbb{R}}$ be convex with proper domain D. Assume X is an integrable random variable such that $\Phi(X)$ is integrable. Then $\Phi\left(\mathbb{E}(X)\right) \leq \mathbb{E}(\Phi \circ X)$. In fact, for each $x_o \in D$ there is an affine function such that $\Phi(x_0) + b(x_o)(x - x_0) \leq \Phi(x)$, $x \in \mathbb{R}$. It follows that $\Phi(x_0) + b(x_o)(\mathbb{E}(X) - x_0) \leq \mathbb{E}(\Phi \circ X)$. In particular, Jensen inequality follows if $x_0 = \mathbb{E}(X)$.

Inequalities II

- Fenchel's inequality: Given the convex function Φ, there exists a convex function Ψ such that $\Psi(y) = \sup_x (xy - \Phi(x))$. In particular, the inequality $xy \leq \Phi(x) + \Psi(y)$ holds for all x, y. It follows that $\mathbb{E}(XY) \leq \mathbb{E}(\Phi \circ X) + \mathbb{E}(\Psi \circ Y)$ if all terms are well defined.
- An important example of Fenchel inequality follows from $xy \leq \frac{1}{\alpha}|x|^\alpha + \frac{1}{\beta}|y|^\beta$, where $\alpha, \beta > 1$ and $\alpha^{-1} + \beta^{-1} = 1$. The integral inequality is $\mathbb{E}(XY) \leq \frac{1}{\alpha}\mathbb{E}(|X|^\alpha) + \frac{1}{\beta}\mathbb{E}\left(|Y|^\beta\right)$.
- For $x \in \mathbb{R}$, $q > 0$ and have $xq \leq \mathrm{e}^x - 1 + q \log q$. If f is a random variable and q is a probability density w.r.t. μ, then $\int fq \, d\mu \leq \int \mathrm{e}^f \, d\mu - 1 + \int q \log q \, d\mu$.
- Lebesgue space: For each $\alpha \geq 1$, define

$$\mathcal{L}^\alpha = \{X \in \mathcal{L} \mid \mathbb{E}(|X|^\alpha) < \infty\} \ .$$

Define

$$X \mapsto (\mathbb{E}(|X|^\alpha))^{1/\alpha} = \|X\|_\alpha$$

.

Inequalities III

- Hölder inequality. Apply Fenchel inequality to $f = X / \|X\|_\alpha$ and $g = Y / \|Y\|_\beta$. It follows

$$\mathbb{E}(fg) = \mathbb{E}\left(\frac{X}{\|X\|_\alpha}\frac{Y}{\|Y\|_\beta}\right) \leq \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

  It follows that $\mathbb{E}(XY) \leq \|X\|_\alpha \|Y\|_\beta$.

- Minkowski inequality. Apply Hölder inequality to

$$\mathbb{E}(|X+Y|^\alpha) = \mathbb{E}\left(|X+Y|\,|X+Y|^{\alpha-1}\right) \leq \mathbb{E}\left(|X|\,|X+Y|^{\alpha-1}\right) + \mathbb{E}\left(|Y|\,|X+Y|^{\alpha-1}\right)$$

  to get $\|X+Y\|_\alpha \leq \|X\|_\alpha + \|Y\|_\alpha$.

Change of variable formula

Let be given a measure space $(S, \mathcal{F}, \mu)$, a measurable space $(\mathbb{X}, \mathcal{G})$ and a measurable mapping ϕ: $S \to \mathbb{X}$, $p^{-1}: \mathcal{G} \to \mathcal{F}$. Let $\phi_{\#}\mu = \mu \circ \phi^{-1}$ be the push-forward measure.

- If $h \in \mathcal{S}(\mathbb{X}, \mathcal{G})$ i.e. $h = \sum_{k=1}^{n} b_k \mathbf{1}_{B_k}$, $b_k \in \mathbb{R}$, $B_k \in \mathcal{G}$, $k = 1, \ldots, n$. Then

$$\int h \; d\phi_{\#}\mu = \sum_{k=1}^{n} b_k \phi_{\#}\mu(B_k) = \sum_{k=1}^{n} b_k \mu[\phi^{-1}(B_k)] =$$

$$\sum_{k=1}^{n} b_k \int \mathbf{1}_{B_k} \circ \phi \; d\mu = \int h \circ \phi \; d\mu$$

- If $f \in \mathcal{L}_+$, then MON implies $\int f \; d\phi_{\#}\mu = \int f \circ \phi \; d\mu$
- If $f : \mathbb{X} \to \mathbb{R}$ is measurable and $f \circ \phi \in \mathcal{L}^1(S, \mathcal{F}, \mu)$ then $f \in \mathcal{L}^1(\mathbb{X}, \mathcal{G}, \phi_{\#}\mu)$ and $\int f \; d\phi_{\#}\mu = \int f \circ \phi \; d\mu$.

Product measure I

Definition

Given measure spaces $(S_i, \mathcal{F}_i, \mu_i)$, $i = 1, \dots, n$, $n = 2, 3, \dots$, the product measure space is

$$(S, \mathcal{F}, \mu) = \otimes_{i=1}^n (S_i, \mathcal{F}_i, \mu_i) = (\times_{i=1}^n S_i, \otimes_{i=1}^n \mathcal{F}_i, \otimes_{i=1}^n \mu_i),$$

where

$$\mathcal{F} = \otimes_{i=1}^n \mathcal{F}_i = \sigma \left\{ \times_{i=1}^n A_i \,|\, A_i \in \mathcal{F}_i, i = 1, \dots, n \right\}$$

and $\mu = \otimes_{i=1}^n \mu_i$ is the unique measure on $(\times_{i=1}^n S_i, \otimes_{i=1}^n \mathcal{F}_i)$ such that

$$\mu(\times_{i=1}^n A_i) = \prod_{i=1}^n \mu_i(A_i), \quad A_i \in \mathcal{F}_i, i = 1, \dots, n.$$

- Let $X_i \colon S \mapsto S_i$, $i = 1, \dots, n$, be the projections. Then $\otimes_{i=1}^n \mathcal{F}_i = \sigma \left\{ X_i | i = 1, \dots, n \right\}$.
- Examples: Counting measure on $\mathbb{N}^2$, Lebesgue measure on $\mathbb{R}^2$, the finite Bernoulli scheme.
- Product measure of probability measures is a probability measure.

Product measure II

Recall all measures are σ-finite. Assume $n = 2$.

Sections

If $C \in= \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, then for each $x_1 \in S_1$ the set $\{x_2 \in S_2 \,|\, (x_1, x_2) \in C\}$ belongs to $\mathcal{F}_2$.

Proof.

Let $\mathcal{O}$ be the family of all subsets of S for which the proposition is true. $\mathcal{O}$ is a σ-algebra that contains all the measurable rectangles, hence $\mathcal{F} \subset \mathcal{O}$. □

Partial integration

The mapping $S_1 \colon x_1 \mapsto \mu_2 \{x_2 \in S_2 \,|\, (x_1, x_2) \in C\}$ is non-negative and $\mathcal{F}_1$-measurable.

Proof.

If $C \in \mathcal{F}$ then the function is well defined. The set of all $C \in \mathcal{F}$ such that the function is measurable contains measurable rectangles, is a π-system, and is a d-system. □

Product measure III

Product measure: existence

The set function $\mu\colon \mathcal{F} \ni C \mapsto \int \mu_2 \{x_2 \in S_2 \,|\, (x_1, x_2) \in C\} \;\; \mu_1(dx-1)$ is a measure such that $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ on measurable rectangles. Hence, $\mu = \mu_1 \otimes \mu_2$.

Proof.

The integral exists because the integrand is non-negative. $\mu(\emptyset) = 0$; if $(C_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$ of disjoint events, then for all $x_1 \in S_1$ we have

$$\mu_2 \{x_2 \in S_2 \,|\, (x_1, x_2) \in \cup_{n\in\mathbb{N}} C_n\} = \mu_2 \left(\cup_{n\in\mathbb{N}} \{x_2 \in S_2 \,|\, (x_1, x_2) \in C_n\}\right) = \sum_{n\in\mathbb{N}} \mu_2 \{x_2 \in S_2 \,|\, (x_1, x_2) \in C_n\}$$

MON implies

$$\mu\left(\cup_{n\in\mathbb{N}} C_n\right) = \int \sum_{n\in\mathbb{N}} \mu_2 \{x_2 \in S_2 \,|\, (x_1, x_2) \in C_n\} \;\; \mu_1(dx_1) = \sum_{n\in\mathbb{N}} \int \mu_2 \{x_2 \in S_2 \,|\, (x_1, x_2) \in C_n\} \;\; \mu_1(dx_1) = \sum_{n\in\mathbb{N}} \mu(C_n)$$

Product measure IV

- Consider $n = 3$. The product measure space

$$\otimes_{i=1}^{3}(S_i, \mathcal{F}_i, \mu_i) = (\times_{i=1}^{3} S_i, \otimes_{i=1}^{3}\mathcal{F}_i, \otimes_{i=1}^{3}\mu_i)$$

  is identified with

$$(S_1 \times S_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2) \otimes (S_3, \mathcal{F}_3, (\mu_1 \otimes \mu_2) \otimes \mu_3)$$

  One has to check that

$$(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3 = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$$

- The $n = \infty$ case requires Charateodory. See the Bernoulli scheme example.

Fubini theorem I

Section

Let $f: S_1 \times S_2 \to \mathbb{R}$ be $\mathcal{F}_1 \otimes \mathcal{F}_2$ measurable. For all $x_1 \in S_1$ the function $f_{x_1}: x_2 \mapsto f(x_1, x_2)$ is $\mathcal{F}_2$-measurable.

Proof.

For each $y \in \mathbb{R}$, consider the level set
$C = \{(x_1, x_2) \,|\, f(x_1, x_2) \leq y\} \in \mathcal{F}_1 \otimes \mathcal{F}_2$. The set $\{(x_2) \,|\, f x_1(x_2) \leq y\}$ is the x_1-section of C. $\square$

Theorem (Non-negative integrand)

Let $f: S_1 \times S_2 \to \mathbb{R}$ be $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable and non-negative. Then the mapping $S_1 \ni x_1 \mapsto \int f(x_1, x_2)\ \mu_2(dx_2)$ is $\mathcal{F}_1$-measurable and

$$\int f\ d\mu_1 \otimes \mu_2 = \int \left(\int f(x_1, x_2)\ \mu_2(dx_2) \right)\ \mu_1(dx_1)$$

§8-5 of Malliavin.

Fubini theorem II

Theorem (Integrable integrand)

Let $f : S_1 \times S_2 \to \mathbb{R}$ be $\mu_1 \otimes \mu_2$-integrable. Then the mapping $S_1 \ni x_1 \mapsto \int f(x_1, x_2)\ \mu_2(dx_2)$ is μ_1-integrable and

$$\int f\ d\mu_1 \otimes \mu_2 = \int \left(\int f(x_1, x_2)\ \mu_2(dx_2) \right)\ \mu_1(dx_1)$$

Proof: Non-negative integrand.

Choose an increasing sequence of simple non-negative functions converging to f and use MON. □

Proof: Integrable integrand.

Decompose $f = f^+ - f^-$ and use the previous form of the theorem. □

Independence

Definition

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

1. The sub-σ-algebras $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are independent if $A_i \in \mathcal{F}_i$, $i = 1, \ldots, n$, implies $\mu(A_1 \cap \cdots \cap A_n) = \mu(A_1) \cdots \mu(A_n)$.
2. The random variables $X_i \colon \Omega \to S_i$, $X_i^{-1} \colon \mathcal{G}_i \to \mathcal{F}$, $i = 1, \ldots, n$, are independent, if

$$(X_1, \ldots, X_n)_{\#}\mu = (X_1)_{\#}\mu \otimes \cdots \otimes (X_n)_{\#}\mu$$

If $\mathcal{F}_i = \sigma(X_i)$, the 1. and 2. are equivalent. If $A_i = X_i^{-1}(B_i)$, $i = 1, \ldots, n$,

$$\begin{aligned}
&\mu(A_1 \cap \cdots \cap A_n) = \mu(X_1^{-1}(B_1) \cap \cdots \cap X_n^{-1}(B_n)) = \\
&\mu((X_1, \ldots, X_n)^{-1}(B_1 \times \cdots \times B_n)) = (X_1, \ldots, X_n)_{\#}\mu(B_1 \times \cdots \times B_n) = \\
&(X_1)_{\#}\mu \otimes \cdots \otimes (X_n)_{\#}\mu(B_1 \times \cdots \times B_n) = (X_1)_{\#}\mu(B_1) \cdots (X_n)_{\#}\mu(B_n) = \\
&\mu(X_1^{-1}(B_1)) \cdots \mu(X_n^{-1}(B_n)) = \mu(A_1) \cdots \mu(A_n)
\end{aligned}$$