# Probability 2017
# 5

Giovanni Pistone

DE CASTRO STATISTICS Collegio Carlo Alberto

April 10, 2017; revised

# Martingale convergence: example I

- On the probability space $(\Omega, \mathcal{F}, \mu)$ let $X_1, X_2, \dots$ be IID with $\mathrm{P}(X_1 = 0) = \mathrm{P}(X_1 = 2) = 1/2$.
- Define $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n = 1, 2, \dots$. Because of $\mathrm{E}(Y_n) = 1$ and the independence, $Y_n = \prod_{i=1}^n X_1$ is a martingale for the filtration $(F_n)_{n\mathbb{N}}$.
- Define $\Omega_0 = \{X_n = 2 \colon n \in \mathbb{N}\}$, so that $\mathrm{P}(\Omega_0) = 0$. For each $\omega \notin \Omega_0$ the sequence $Y_n(\omega)$, $n = 1, 2, \dots$ is eventually 0, hence $\lim_{n\to\infty} Y_n = Y_\infty = 0$ a.s.
- However,

$$\lim_{n\to\infty} \mathrm{E}(|Y_n - Y_\infty|) = \lim_{n\to\infty} \mathrm{E}(Y_n) = 1 \neq 0 \ .$$

# Martingale convergence: example II

- On the filtered probability space $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t \in \mathbb{Z}_\geq})$, consider a square-integrable random variable $X$ and the martingale $X_t = \mathrm{E}(X|\mathcal{F}_t)$. It follows from Jensen inequality

$$\mathrm{E}\left(X_t^2\right) = \mathrm{E}\left(\mathrm{E}\left(X|\mathcal{F}_t\right)^2\right) \leq \mathrm{E}\left(\mathrm{E}\left(X^2\middle|\mathcal{F}_t\right)\right) = \mathrm{E}\left(X^2\right) \ .$$

- We have

$$\mathrm{E}\left(X_t^2\right) = \mathrm{E}\left(X_0^2\right) + \sum_{s=1}^{t} \mathrm{E}\left((X_s - X_{s-1})^2\right) \leq \mathrm{E}\left(X^2\right) \ ,$$

  hence the positive series $\sum_{s=1}^{\infty} \mathrm{E}\left((X_s - X_{s-1})^2\right)$ is convergent.

- For $t_1 < t_2$ we have

$$\mathrm{E}\left((X_{t_2} - X_{t_1})^2\right) = \sum_{s=t_1+1}^{t_2} \mathrm{E}\left((X_s - X_{s-1})^2\right)$$

  and $\lim_{n\to\infty} X_t = X_\infty$ in $L^2$-norm exists.



- Ch 12 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

# Martingale convergence: example II - bis

- We have

$$\mathsf{E}\left((\mathsf{E}\left(X_\infty - X|\mathcal{F}_t\right))^2\right) = \mathsf{E}\left((\mathsf{E}\left(X_\infty|\mathcal{F}_t\right) - X_t)^2\right) = \mathsf{E}\left((\mathsf{E}\left(X_\infty - X_t|\mathcal{F}_t\right))^2\right) \leq \mathsf{E}\left((X_\infty - X_t)^2\right) \to 0$$

  hence $\mathsf{E}\left(X_\infty - X|\mathcal{F}_t\right) = 0$ and $\mathsf{E}\left(X_\infty|\mathcal{F}_t\right) = X_t$.

- Let $\mathcal{F}_\infty = \sigma(\mathcal{F}_t \colon t \in \mathbb{Z}_\geq)$. $\mathcal{F}_\infty$ is generated by the $\pi$-system $\cup_{t\in\mathbb{Z}_\geq}\mathcal{F}_t$. It follows that $X_\infty = \mathsf{E}\left(X|\mathcal{F}_\infty\right)$.

- It is possible to prove with Borel-Cantelli lemma that $\lim_{n\to\infty} X_t = X_\infty$ a.s.

# Uniform integrability

- If $\mathrm{E}(|X|) < \infty$, then $\lim_{c\to\infty} \mathrm{E}\left(\mathbf{1}_{|X|>c}|X|\right) = 0$ by monotone convergence.

## Definition

The sequence $(X_t)_{t\in\mathbb{N}}$ is uniformly integrable if

$$\lim_{c\to\infty} \sup\left\{\mathrm{E}\left(\mathbf{1}_{|X|>c}|X_t|\right)\middle| t\in\mathbb{N}\right\} = 0\ .$$

- Example I is not uniformly integrable. $\mathrm{E}(\mathbf{1}_{Y_n>c}Y_n)$ is zero if $2^n \leq C$, otherwise is equal to $Y_n$.
- Example II is uniformly integrable.

$$\mathrm{E}\left(\mathbf{1}_{|X_t|>c}|X_t|\right) \leq c^{-1}\,\mathrm{E}\left(X_t^2\right) \leq c^{-1}\,\mathrm{E}\left(X^2\right)$$

# Uniform integrability and convergence

## Theorem

*If $\lim_{n\to\infty} X_n = X$ a.s., then the limit holds il $L^1$ if, and only if, the sequence is uniformly integrable.*

- See Ch 13 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

# Upcrossing

- Consider the discrete trajectory $x\colon 0, 1, \ldots, N \to \mathbb{R}$ and fix two real numbers $a < b$. There is an upcrossing if there exists two times $s < t$ such that $x(s) < a$ and $b < x(t)$.
- Let us find the maximum number $U$ of upcrossing as follows:
  - wait until the first time $s_1$ when the the trajectory comes below $a$ i.e.
$$s_1 = \inf \{s > 0 | x(s) < a\} \ ;$$
  - wait until the first time when the trajectory comes above $b$ i.e.,
$$t_1 = \inf \{s > s_1 | x(s) > b\} \ ;$$
  - start again with
$$s_2 = \inf \{s > t_1 | x(s) < a\} \ ;$$
  - and so on.
- The number of upcrossing $U_N[a, b]$ is the greatest integer $u$ such that $t_u < +\infty$. It is finite.

# Upcrossing inequality

- Let $u = U_N[a,b]$. For each $1 \leq j \leq u$ we have $x(t_j) - x(s_j) > (b-a)$, hence

$$\sum_{j=1}^{u} (x(t_j) - x(s_j)) > (b-a)u$$

- Let us complete the previous sum to a possible $s_{u+1} < N$ adding the corresponding increment

$$\sum_{j=1}^{u} (x(t_j) - x(s_j)) + (x(N) - x(s_{u+1}) > (b-a)u - (x(N) - a)^-$$

- If the trajectory is infinite, $x \colon \mathbb{Z}_{\geq} \to \mathbb{R}$, we can define an infinite increasing sequence in $\mathbb{Z}_{\geq} \cup \{\infty\}$ such that for each $N \in \mathbb{N}$ we have

$$\sum_{j=1}^{\infty} (x(t_j \wedge N) - x(s_j \wedge N)) > (b-a)U_N[a,b] - (x(N) - a)^-$$

- $\lim_{N \to \infty} U_N[a,b] = U_\infty[a,b]$ is the (possibly infinite) total number of upcrossings.

# Upcrossings of an adapted process

- Let $(X)_{t \in \mathbb{Z}_\geq}$ be a real process adapted to $(\mathcal{F}_t)_{t \in \mathbb{Z}_\geq}$. If $A$ is a stopping time and $I$ a real interval, then

$$B = \inf \{t \geq A | X_t \in I\}$$

is a stopping time. In fact

$$\{B = t\} = \cup_{s=0}^{t} \{A = s, B = t\} = \\ \cup_{s=0}^{t} \{A = s\} \cap \{X_t \in I\} \cap_{u=s}^{t-1} \{X_u \notin I\} \in \mathcal{F}_t \ .$$

- All $S_j \wedge N$, $T_j \wedge N$, $j \in \mathbb{N}$ are stopping times and

$$\sum_{j=1}^{\infty} \mathrm{E}\left(X_{t_j \wedge N} - X_{s_j \wedge N}\right) + \mathrm{E}\left((X_N - a)^-\right) > (b-a)\,\mathrm{E}\left(U_N[a,b]\right)$$

# Doob a.s. convergence

## Theorem

*Let* $(X)_{t\in\mathbb{Z}_\geq}$ *be an integrable real process adapted to* $(\mathcal{F}_t)_{t\in\mathbb{Z}_\geq}$ *such that*

1. *For each couple* $A \leq B$ *of bounded stopping times it holds* $\operatorname{E}(X_B - X_A) \leq 0$ *i.e., the process is a supermartingale, and*
2. $\sup_{t\in\mathbb{N}} \operatorname{E}(|X_t|) < \infty$.

*Then*

$$\operatorname{E}(U_\infty[a,b]) \leq |a| + \sup_{t\in\mathbb{N}} \operatorname{E}(|X_t|) \ .$$

*In particular, for all* $a, b$ *the number of upcrossing* $U_\infty[a,b]$ *is finite a.s. so that* $\lim_{n\to\infty} X_t$ *exists a.s.*

## Proof.

A real sequence $(x(t))_{t\in\mathbb{Z}_\geq}$ is confergent to a finite limit if, and only if, the number of upcrossings is finite for each $a, b \in \mathbb{Q}$ (assumption 1) and $\liminf_{t\to\infty} |x(t)| \neq +\infty$ (assumption 2). $\square$