Probability 2017
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Giovanni Pistone
www.giannidiorestino.it

DE CASTRO STATISTICS
Collegio Carlo Alberto

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Probability space

Definition

A *probability space* is a triple $(\Omega, \mathcal{F}, \mathrm{P})$ of a sample space Ω (set of possible worlds), a σ-algebra $\mathcal{F}$ on Ω, a probability measure $\mathrm{P} \colon \mathcal{F} \to [0,1]$. An element $\omega \in \Omega$ is a sample point (world); an element $A \in \mathcal{F}$ is an event; the value $\mathrm{P}(A)$ is the probability of the event A.

- Examples: a finite set, all its subsets, a probability function $p \colon \Omega \to \mathbb{R}_{>0}$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$; $\mathbb{Z}_{\geq}$ with all its subsets, and a probability function $p \colon \mathbb{Z}_{\geq} \to \mathbb{R}_{>0}$ such that $\sum_{k=0}^{\infty} p(k) = 1$; the restriction of a probability space to a sub-σ-algebra; the product of two probability spaces.
- Bernoulli trials. Let $\Omega = \{0,1\}^{\mathbb{N}}$ and let $\mathcal{F}_n = \left\{ A \times \{0,1\} \times \{0,1\} \times \cdots \middle| A \subset \{0,1\}^n \right\}$, $\mathcal{F} = \sigma(\mathcal{F}_n \colon n \in \mathbb{N})$. Given $\theta \in [0,1]$, the function $p_n(x_1 x_2 \cdots x_n \cdots) = \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n - \sum_{i=1}^n x_i}$ uniquely defines probability spaces $(\Omega, \mathcal{F}_n, \mathrm{P}_n)$, $n \in \mathbb{N}$, such that $\mathrm{P}_{n+1}|_{\mathcal{F}_n} = \mathrm{P}_n$, hence a probability measure P on $\mathcal{F}$.

Ch. 2 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

$\limsup$ and $\liminf$

Definition

- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

$$\limsup_{n\to\infty} a_n = \wedge_{m\in\mathbb{N}} \vee_{n\geq m} a_n \quad \text{(maximum limit)}$$

$$\liminf_{n\to\infty} a_n = \vee_{m\in\mathbb{N}} \wedge_{n\geq m} a_n \quad \text{(minimum limit)}$$

- Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of events in the measurable space $(\Omega, \mathcal{F})$.

$$\limsup_{n\to\infty} E_n = \cap_{m\in\mathbb{N}} \cup_{n\geq m} E_n \quad (E_n \text{ infinitely often})$$

$$\liminf_{n\to\infty} E_n = \cup_{m\in\mathbb{N}} \cap_{n\geq m} E_n \quad (E_n \text{ eventually})$$

Theorem (Fatou lemma)

$$\mathrm{P}\left(\liminf_{n\to\infty} E_n\right) \leq \liminf_{n\to\infty} \mathrm{P}\left(E_n\right) \leq \limsup_{n\to\infty} \mathrm{P}\left(E_n\right) \leq \mathrm{P}\left(\limsup_{n\to\infty} E_n\right)$$

About Fatou lemma

- Notice that $(\limsup_n E_n)^c = \liminf_n E_n^c$.
- Notice that $\limsup_n \mathbf{1}_{E_n} = \mathbf{1}_{\limsup_n E_n}$.
- Proof of FL. Write $\cap_m \cup_{n \geq m} E_n = \cap_m G_m$ so that $G_m \downarrow G = \limsup_n E_n$. We have $\mathrm{P}(G_m) \geq \vee_{n \geq m} \mathrm{P}(E_n)$; monotone continuity implies $\mathrm{P}(G_m) \downarrow \mathrm{P}(G)$ hence, $\wedge_m \mathrm{P}(G_m) = \mathrm{P}(G)$.
- BC1. Assume $\sum_{n=1}^{\infty} \mathrm{P}(E_n) < +\infty$. We have for all $m \in \mathbb{N}$ that

$$\mathrm{P}\left(\limsup_n E_n\right) \leq \mathrm{P}(\cup_{n \geq m} E_n) \leq \sum_{n=m}^{\infty} \mathrm{P}(E_n) \to 0 \quad \text{if } m \to \infty$$

hence $\mathrm{P}(\limsup_n E_n) = 0$.

Measurable function

Definition

Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 1, 2$, we say that the function $h\colon S_1 \to S_2$ is measurable, or is a random variable, if for all $B \in \mathcal{S}_2$ the set $h^{-1}(B) = \{s \in S_1 | h(s) \in B\}$ belongs into $\mathcal{S}_1$.

Theorem

- *Let $\mathcal{C} \subset \mathcal{S}_2$ and $\sigma(C) = \mathcal{S}_2$. If $h^{-1}\colon \mathcal{C} \to \mathcal{S}_1$, then h is measurable.*
- *Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 1, 2, 3$, if both $h\colon S_1 \to S_2$, $g : S_2 \to S_3$ are measurable functions, then $g \circ f : S_1 \to S_3$ is a measurable function.*
- *Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 0, 1, 2$ and $h_i\colon S_0 \to S_j$, $j = 1, 2$, consider $h = (h_1, h_2)\colon S_0 \to S_1 \times S_2$. with product space $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$, Then both h_1 and h_2 are measurable if, and only if, h is measurable.*

- Ch. 3 of Ch. 2 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.

Image measure

Definition

Given measurable spaces $(S_i, \mathcal{S}_i)$, $i = 1, 2$, a measurable function $h\colon S_1 \to S_2$, and a measure μ_1 on $(S_1, \mathcal{S}_1)$, then $\mu_2 = \mu_1 \circ h^{-1}$ is a measure on $(S_2, \mathcal{S}_2)$. We write $h_{\#}\mu_1 = \mu_2 \circ h^{-1}$ and call it image measure. If μ_1 is a probability measure, we say that $h_{\#}\mu_1$ is the distribution of the random variable h.

- Bernoulli scheme Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the Bernoulli scheme, and define $X_t\colon \Omega \to \{0, 1\}$ to be the t-projection, $X_t(x_1 x_2 \cdots) = x_t$. It is a random variable with Bernoulli distribution $\mathrm{B}(\theta)$. The random variable $Y_n = X_1 + \cdots + X_n$ has distribution $\mathrm{Bin}(\theta, n)$. The random variable $T = \inf\{k \in \mathbb{N} | X_k = 1\}$ has distribution $\mathrm{Geo}(\theta)$.

Real random variable

Definition

Let $(S, \mathcal{S})$ be a measurable space. A real random variable is a real function $h \colon S \to \mathbb{R}$ with is measurable into $(\mathbb{R}, \mathcal{B})$.

Theorem

- *$h \colon S \to \mathbb{R}$ is a real random variable if, and only if, for all $c \in \mathbb{R}$ the level set $\{s \in S\}\, h(s) \leq c$ is measurable. The same property holds with $\leq$ replaced by $<$ or $\geq$ or $>$. The condition can be taken as a defintion of extended random variable i.e.* $h \colon S \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.
- *If $g, h \colon S \to \mathbb{R}$ are real random variables and $\Phi \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, then $\Phi \circ (g, h)$ is a real random variable.*
- *Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of real random variables on $(S, \mathcal{S})$. Then* $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ *are real random variable.*

A monotone-class theorem

Theorem

Let $\mathcal{H}$ be a vector space of bounded real functions of a set S and assume $\mathbf{1} \in \mathcal{H}$. Assume

1. *$\mathcal{H}$ is a monotone class i.e., if for each bounded increasing sequence $(f_n)_n \in \mathbb{N}$ in $\mathcal{H}$ the function $\vee_n f_n$ belong to $\mathcal{H}$.*
2. *$\mathcal{H}$ contains the indicator functions of a π-system $\mathcal{I}$.*

Then, $\mathcal{H}$ contains all bounded measurable functions of $(S, \sigma(I))$.

- Application. Consider measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$. Define $\Omega = \Omega_1 \times \Omega_2$ and $\mathcal{I} = \{A_1 \times A_2 | A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$. Then $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{I})$. Let $\mathcal{H}$ be the set of all bounded real funtions $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ such that for each fixed $x \in \Omega_1$ the mapping $\Omega_2 \ni y \mapsto f(x, y)$ is $\mathcal{F}_2$-measurable and for each fixed $y \in \Omega_2$ the mapping $\Omega_1 \ni x \mapsto f(x, y)$ is $\mathcal{F}_1$-measurable.

- §3.14 and §A3.1 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991