Probability 2017
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Measurable space

Definition

- A family $\mathcal{B}$ of subsets of S is an algebra on S if it contains $\emptyset$ and S, and it is stable for the complements, finite unions, and finite intersection.
- A family $\mathcal{F}$ of subsets of S is a σ-algebra on S if it is an algebra on S and it is stable for denumerable unions and intersections.
- A measurable space is a couple $(S, \mathcal{F})$, where S is a set and $\mathcal{F}$ is a σ-algebra on S.

- Given the family $\mathcal{C}$ of subsets of S, the σ-algebra generated by $\mathcal{C}$ is $\sigma(\mathcal{C}) = \cap \{\mathcal{A} | \mathcal{C} \subset \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra$\}$.
- Examples: the algebra generated by a finite partition; the algebra generated on a finite set S by a function $f : S$; the Borel σ-algebra of $\mathbb{R}$ is generated by the open intervals, or by the closed intervals, or by the intervals, or by the open sets, or by semi-infinite intervals.

§1.1 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

Measure space

Definition

- A measure μ of the measurable space $(S, \mathcal{F})$ is a mapping $\mu\colon \mathcal{F} \to [0, +\infty]$ such that $\mu(\emptyset) = 0$ and for each sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint elements of $\mathcal{F}$, $\mu(\cup_{n\in\mathbb{N}} A_n) = \sum_{i=1}^{\infty} \mu(A_n)$.
- A measure is finite if $\mu(S) < +\infty$; a measure is σ-finite if there is a sequence $(S_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\cup_{n\in\mathbb{N}} S_n = S$ and $\mu(S_n) < +\infty$ for all $n \in \mathbb{N}$.
- A probability measure is a finite measure such that $\mu(S) = 1$; a probability space is the triple $(S, \mathcal{F}, \mu)$, where μ is a probability measure.

- Examples: probability measure on a partition; probability measure on a denumerable set.
- Equivalently, a probability measure is finitely additive and sequentially continuous at $\emptyset$

§1.3-5 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

π-system

Definition

Let S be a set. A π-system on S is a family $\mathcal{I}$ of subsets of S which is stable under finite intersection.

- Examples: the family of all points of a finite set and the empty set; the family of open intervals of $\mathbb{R}$; the famililiy of closed intervals of $\mathbb{R}$; the family of cadlàg intervals of $\mathbb{R}$; the family of convex (resp. open convex, closed convex) subsets of $\mathbb{R}^2$; the family of open (resp. closed) set in a topological space.
- If $\mathcal{I}_i$ is a π-system of S_i, $i = 1, \dots, n$, then $\{\times_{i=1}^n I_i | I_i \in \mathcal{I}_i\}$ is a π-system of $\times_{i=1}^n S_i$.
- The family of all real functions of the form $\alpha_0 + \sum_{j=1}^n \alpha_j \mathbf{1}_{I_i}$, $n \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, $j = 0, \dots, n$ is a vector space and it is stable for multiplication.

§1.6 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

d-system

Definition

Let S be a set. A d-system on S is a family $\mathcal{D}$ of subsets of S such that

1. $S \in \mathcal{D}$
2. If $A, B \in \mathcal{D}$ and $A \subset B$, then $B \setminus A \in \mathcal{D}$. (Notice that $S \setminus A = A^c$)
3. If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{D}$, then $\cup_{n\in\mathbb{N}} \in \mathcal{D}$

- Given probabilities μ_i and $i = 1, 2$ on the measurable space $(S, \mathcal{F})$, the family $\mathcal{D} = \{A \in \mathcal{F} | \mu_1(A) = \mu_2(A)\}$ in a d-system.
- Given measurable spaces $(S_i, \mathcal{F}_i)$, $i = 1, 2$, the product space $(S, \mathcal{F}) = (S_1 \times S_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma\{A_1 \times A_2 | A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$, and $x \in S_1$, the family $\mathcal{D} = \{A \in \mathcal{F}_1 \otimes \mathcal{F}_2 | A \cap \{x\} \times S_2 = \{x\} \times A_x, A_x \in \mathcal{F}_2\}$ is a d-system.

§A1.2 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991

Dynkin's lemma

Theorem

1. *A family of subsets of S is a σ-algebra if, and only if, it is both a d-system and a π-system.*
2. *If $\mathcal{I}$ is a π-system, then $d(\mathcal{I}) = \sigma(\mathcal{I})$.*
3. *Any d-system that contains a π-system contains the σ-algebra generated by the π-system.*

Theorem

If two probability measures on the same measurable space agree on a π-system they are equal.

§A1.3 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991