# Stochastic Processes 2014

# 4. Introduction to Lévy Process and Stochastic Analysis


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# Plan

1. Poisson Process (Formal construction)
2. Wiener Process (Formal construction)
3. Infinitely divisible distributions (Lévy-Khinchin formula)
4. Lévy processes (Generalities)
5. Stochastic analysis of Lévy processes (Generalities)

# References


Applebaum David Applebaum, *Lévy processes and stochastic calculus*, second ed., Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, 2009

Iacus Stefano M. Iacus, *Option pricing and estimation of finantial models with R*, Wiley, 2011

Sato Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 2013, Translated from the 1990 Japanese original, Revised edition of the 1999 English translation

Williams David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991.


# 1. CÀDLÀG trajectory

- A trajectory $x\colon \mathbb{R}_+ \to \mathbb{R}$ is *right continuous and with left limits (CÀDLÀG )* if

$$\lim_{h\downarrow 0} x(t+h) = x(t), \qquad \lim_{h,k\downarrow 0} (x(t-h) - x(t-k)) = 0.$$

  We write

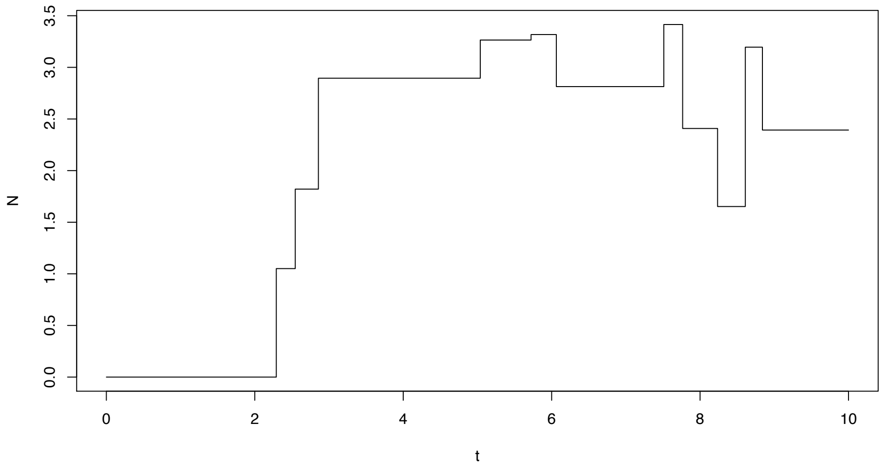
$$\lim_{h\downarrow 0} x(t-h) = x(t-), \quad \Delta x(t) = x(t) - x(t-).$$

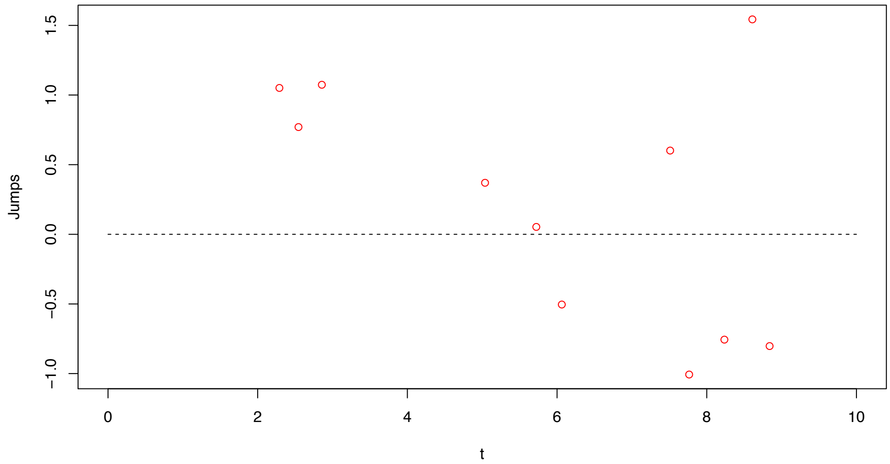
  If $\Delta x(t) \neq 0$, we say that $(t, \Delta x(t))$ is a *jump*. In the couple $(t, \Delta x(t))$, $t$ is the *jump time*, $\Delta x(t)$ is the *marker of the jump*.

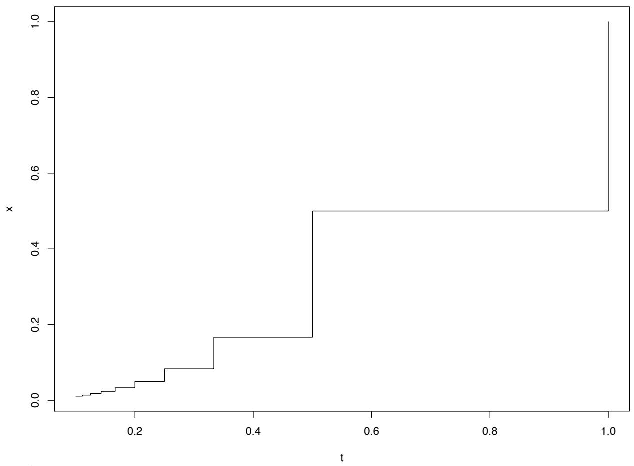
- For any given a finite time interval $[0, T[$ and for each $\epsilon > 0$ the CÀDLÀG trajectory $x$ has a finite number of jumps with $t \in [0, T[$ and $\Delta x(t) > \epsilon$. The set of all jumps is countable.

- The set of jumps is best described as a the $\sigma$-finite measure $\sum \delta_{(t,\Delta x(t))}$ on $\mathbb{R}_+ \times \mathbb{R}_*$, $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$, defined on rectangles by

$$\mu^x(I \times B) = \#\{t \in I | \Delta x(t) \in B\}.$$



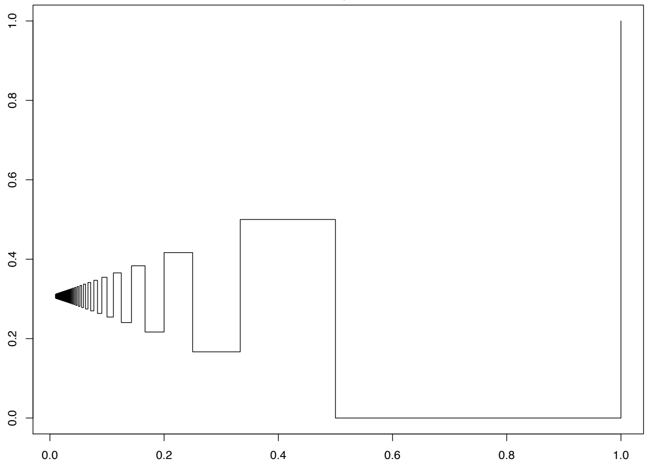




$$x\left(\frac{1}{n}\right) = \frac{1}{n}$$

$$\Delta x\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$\sum \Delta x\left(\frac{1}{n}\right) = 1$$




$$\Delta x\left(\frac{1}{n}\right) = (-1)^n \frac{1}{n}$$

$$\sum \Delta x\left(\frac{1}{n}\right) = \sum \frac{(-1)^{n-1}}{n} < +\infty$$

$$\sum \left|\Delta x\left(\frac{1}{n}\right)\right| = \sum \frac{1}{n} = +\infty$$

# 4. CÀDLÀG trajectory: removing the jumps

- Assume $\int_0^t \int_{\mathbb{R}_*} |y| \ \mu^x(ds, dy) < \infty$ for all $t$. Then the *sum of jumps* $\int_0^t \int_{\mathbb{R}_*} y \ \mu^x(ds, dy)$ is finite for all $t$ and CÀDLÀG .
- In such a case the trajectory $t \mapsto x(t) - \int_0^t \int_{\mathbb{R}_*} y \ \mu^x(ds, dy)$ is continuous.
- If $\int_0^t \int_{\mathbb{R}_*} |y| \ \mu^x(ds, dy) < \infty$ for some $t$, then the sum of jumps is not defined, but the sum of jumps larger then $\epsilon > 0$, $t \mapsto \int_0^t \int_{|y|>\epsilon} y \ \mu^x(ds, dy)$ is finite for all $t$ and CÀDLÀG .
- In such a case the trajectory $t \mapsto x(t) - \int_0^t \int_{|y|>\epsilon} y \ \mu^x(ds, dy)$ is CÀDLÀG with jumps smaller or equal to $\epsilon$ is absolute value.
- Note that this constructions are non-anticipative.
- If $x$ is increasing, then all jumps have a positive marking and the sum of jumps exists.



# 5. CÀDLÀG trajectory: variation

- The *variation* of a CÀDLÀG trajectory $x$ is the increasing function

$$t \mapsto \overline{x}(t) = \lim_{t_0=0<t_1<\cdots<t_n=t} \sum_{i=1}^n |x(t_i) - x(t_{i-1})| \, .$$

- The *quadratic variation* $t \mapsto [x](t)$ of the CÀDLÀG trajectory $x$ is the increasing function

$$t \mapsto [x](t) = \lim_{t_0=0<t_1<\cdots<t_n=t} \sum_{i=1}^n (x(t_i) - x(t_{i-1}))^2$$

- The two variations are related by the inequality

$$\sum_{i=1}^n (x(t_i) - x(t_{i-1}))^2 \leq \max \{|x(t_i) - x(t_{i-1})|\} \, \overline{x}(t).$$

- Note that from $(a-b)^2 = -2b(a-b) + (a^2 - b^2)$ follows

$$[x](t) = x(t)^2 - x(0)^2 - 2 \lim_{t_0=0<t_1<\cdots<t_n=t} \sum_{i=1}^n x(t_{i-1})(x(t_i) - x(t_{i-1})).$$

# 6. Continuous $L^2$ martingales

## Definition

A continuous process $M$ is an $L^2$ martingale if

1. $\mathbb{E}\left(M_t^2\right) < +\infty$, and
2. $\mathbb{E}\left(M_t \,|\mathcal{F}_s\right) = M_s, \quad 0 \leq s < t.$

## Theorem

*Let $M_n$, $n = 1, 2, \ldots$ be a sequence of $L^2$ continuous martingales.Let $T$ be a finite horizon and assume that the $L^2$ limit of $M_n(T)$ exists, i.e. there exists a random variable $M$ such that*

$$\lim_{n \to \infty} \mathbb{E}\left((M_n(T) - M(T))^2\right) = 0.$$

*Then there exist an $L^2$ continuous martingale $M_t$, $t \in [0, T]$ such that*

$$\lim_{n \to \infty} \mathbb{E}\left(sup_{t \in [0,T]}(M_n(t) - M_t)^2\right) = 0.$$

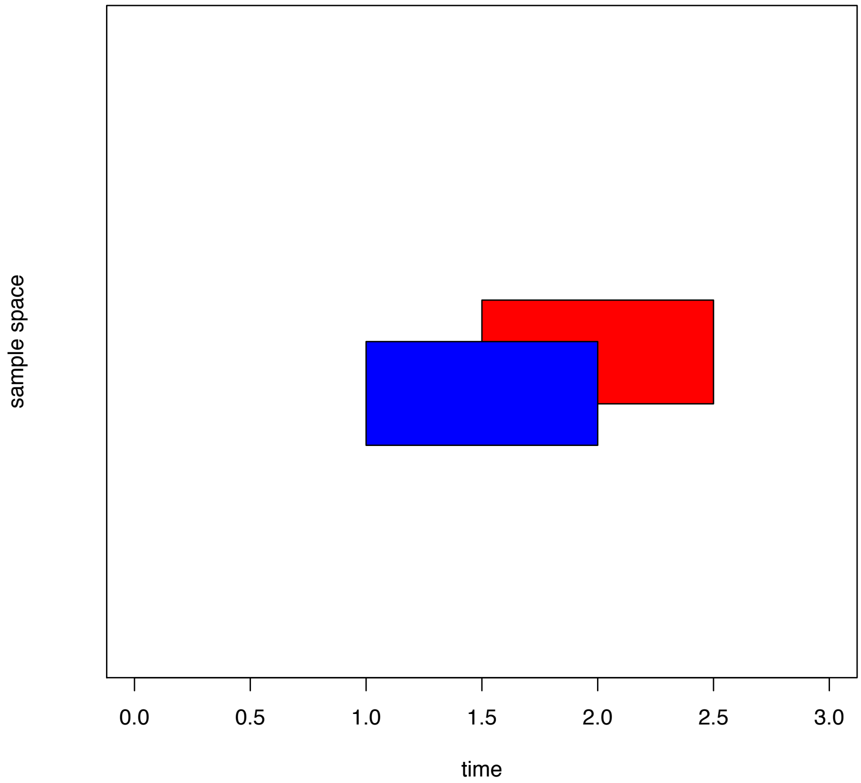
[Williams] 14.11



# 7. Predictable process

A basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t \geq 0}))$ is given.

- A *left continuous predictable interval* is a subset of $\Omega \times \mathbb{R}_+$ of the form $A \times ]a, b]$, with $0 \leq a < b$ and $A \in \mathcal{F}_a$.
- The set of all left continuous predictable intervals is $\pi$-class that generates a $\sigma$-algebra $\mathcal{P}$ called *predictable* $\sigma$-algebra.
- The process $(\omega, t) \mapsto (\omega \in A, a < t \leq b)$ is a left continuous adapted process. Let us call such a process an elementary predictable process. A linear combination of elementary predictable processes is a *simple* predictable process.
- If a process is adapted and left continuous, then it is predictable.




# 9. Stochastic integral: simple integrand

A CÀDLÀG process on the basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ is given.

- If $Y$ is elementary predictable $Y_t(\omega) = (\omega \in A, a < t \leq b)$, define the *stochastic integral* $\int Y \, dX$ to be the process with trajectories

$$t \mapsto (\omega \in A)(X_{min(b,t)} - X_{min(a,t)}), \quad \omega \in \Omega.$$

  We write $\int_0^t Y_s \, dX_s = \left(\int Y \, dx\right)_t$.

- If $\omega \notin A$, then $\left(\int Y \, dX\right)(\omega) = 0$.
- If $\omega \in A$ and $t \leq a$, then $\left(\int Y \, dX\right)_t(\omega) = 0$.
- If $\omega \in A$ and $a \leq t \leq b$, then $\left(\int Y \, dX\right)_t(\omega) = X_t(\omega) - X_a(\omega)$.
- If $\omega \in A$ and $b \leq t$, then $\left(\int Y \, dX\right)_t(\omega) = X_b(\omega) - X_a(\omega)$.
- The definition easily extend to simple predictable process by linearity.

## 10. Ito integral of the simple process $\Delta(t) = Y_1(t_1 \le t)$, $t \ge 0$

- For $\Delta(t) = Y_1(t_1 \le t)$, define the *Ito integral*

$$\int_0^t \Delta(s)dW(s) = \begin{cases} 0 & \text{for } t < t_1 \\ Y_1(W(t) - W(t_1)) & \text{for } t_1 \le t \end{cases}$$
$$= Y_1(W(t) - W(t \wedge t_1))$$

- The Ito integral is a *continuous martingale*

$$\mathbb{E}\left(\int_0^t \Delta(u)dW(u) \middle| \mathcal{F}(s)\right) = \int_0^s \Delta(u)dW(u), \quad s \le t.$$

- The Ito integral is *isometric*

$$\mathbb{E}\left(\left(\int_0^t \Delta(u)dW(u)\right)^2\right) = \mathbb{E}\left(\int_0^t \Delta^2(u)du\right).$$

- The *quadratic variation* of the Ito integral is $\int_0^t \Delta^2(s)ds$.



## 11. Ito integral of a simple process $\Delta$

- For $\Delta(t) = \sum_{j=1}^n Y_j(t_j \le t)$, define the *Ito integral* by linearity. If the interval $[0, t]$ contains the jumps $0 \le t_1 < \cdots t_m \le t$,

$$\int_0^t \Delta(s)dW(s) = \sum_{j=1}^n Y_j(W(t) - W(t \wedge t_j))$$
$$= \sum_{j=1}^m Y_j(W(t) - W(t_j))$$
$$= \sum_{j=1}^m Y_j(\sum_{i=j+1}^m W(t_{i+1}) - W(t_i))$$
$$= \sum_{i=1}^m \Delta(t_i)(W(t_{i+1}) - W(t_i))$$

- The Ito integral is a *continuous martingale*.
- The Ito integral is *isometric*.
- The *quadratic variation* of the Ito integral is $\int_0^t \Delta^2(s)ds$.

# 12. Ito integral of an $L^2$ process

- If $\Delta$ is a process of class $L^2$, there exists a sequence $\Delta_n$, $n = 1, 2, \ldots$ of simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T |\Delta(u) - \Delta_n(u)|^2 \, du\right) = 0.$$

- The Ito integral of a process of class $L^2$ is defined by continuity.
- The Ito integral is a linear operator mapping $L^2$ processes into continuous martingale.
- The Ito integral is isometric.
- The quadratic variation of the Ito integral is

$$\left[\int \Delta dW\right](t) = \int_0^t \Delta^2(u) du$$



# 13. Continuous martingales

If $M$ is a continuous bounded martingale, the computation

$$\begin{aligned} M^2(t) - M^2(s) &= \sum_{j=1}^n M^2(t_j) - M^2(t_{j-1}) \\ &= \sum_{j=1}^n 2M(t_{j-1})(M(t_j) - M(t_{j-1})) + \sum_{j=1}^n (M(t_j) - M(t_{j-1}))^2 \end{aligned}$$

produces the decomposition

$$M^2(t) = M^2(0) + 2\int_0^t M(u) dM(u) + [M](t)$$

and, for an Ito integral,

$$\left(\int_0^t \Delta dW\right)^2 = 2\int_0^t \left(\int_0^s \Delta(u) dW(u)\right) dW(s) + \int_0^t \Delta^2(s) ds$$

# 14. Ito-Doeblin formula

## Definition (Ito process)

An *Ito process* is a process of the form

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds.$$

## Theorem (Ito-Doeblin formula for the Brownian Motion)

- *If $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^2)$ and*
- *$f_x(t, W(t))$, $t \geq 0$, is an $L^2$ process,*

*then $f(t, W(t))$, $t \geq 0$, is an Ito process, and*

$$f(t, W(t)) = f(0, W(0)) + \int_0^t f_t(s, W(s))ds + \int_0^t f_x(s, W(s))dW(s) + \frac{1}{2}\int_0^t f_{xx}(s, W(s))ds.$$



# 15. Proof of Ito-Doeblin formula

We write for $0 \leq s < t \leq T$

$$\begin{aligned}\int_s^t \Delta(u)dW(u) &= \int_0^t \Delta(u)dW(u) - \int_0^s \Delta(u)dW(u)\\ &= \int_0^T (s < u \leq t)\Delta(u)dW(u).\end{aligned}$$

In particular,

$$\begin{aligned}(W(t) - W(s))^2 &= W(t)^2 - W(s)^2 - 2W(s)(W(t) - W(s))\\ &= 2\int_s^t W(u)dW(u) + (t-s) - 2W(s)(W(t) - W(s))\\ &= (t-s) + 2\int_s^t (W(u) - W(s))dW(u)\end{aligned}$$

The Taylor formula of order 1,2 for $f$ gives

$$\begin{aligned}f(t, W(t)) - f(s, W(s)) =& f_t(s, W(s))(t-s)\\ &+ f_x(s, W(s))(W(t) - W(s))\\ &+ \frac{1}{2}f_{xx}(s, W(s))(W(t) - W(s))^2\end{aligned}$$

# 16. Ito-Doeblin formula: Applications

- The process $f(t, W(t))$ is a martingale if $f_{10}(t,x) + \frac{1}{2} f_{02}(t,x) = 0$.
- Let $H_n(x)$ be a polynomial of degree $n$ and define $f(t,x) = t^{n/2} H_n(t^{-1/2}x)$. We have

$$f_{1,0}(t,x) = t^{n/2-1}(\frac{1}{2}H_n(t^{-1/2}x) - \frac{x}{2}H_n'(t^{-1/2}x)),$$
$$f_{02}(t,x) = t^{n/2-1}H_n''(t^{-1/2}x).$$

- The martingale condition is satisfied if

$$nH_n(y) - yH_n'(y) + H_n''(y) = 0.$$

- We can take the *Hermite polynomials*

$$H_n(y) = (-1)^n \mathrm{e}^{\frac{y^2}{2}} \frac{d^n}{dy^n} \mathrm{e}^{-\frac{y^2}{2}}$$

to obtain the *Hermite martingales*

$$M_n(t) = \int_0^t u^{\frac{n}{2}} H_n(u^{-\frac{1}{2}} W(u)) dW(u).$$

[Hint: the $n$-th derivative of $yg(y)$ is $yg^{(n)}(y) + ng^{(n-1)}(y)$]



# 17. Lévy process

The *basis* of process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ is given. The set of trajectories $D(\mathbb{R}_+)$ is the set of continuous to the right, with left limits, functions $x \colon \mathbb{R}_+ \to \mathbb{R}$, CÀDLÀG .

## Definition

$X$ is a *Lévy process* on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ if it is is a CÀDLÀG process, $X \colon \Omega \to D(\mathbb{R}_+)$, such that

1. $X$ is adapted, i.e. $X_t$ is $\mathcal{F}_t$-measurable, $t \geq 0$;
2. $X$ starts from 0, i.e. $X_0 = 0$ a.s.;
3. the increments are independent from the past history, i.e. $(X_t - X_s)$ is independent of $\mathcal{F}_s$, $0 \leq s < t$;
4. the increments are homogeneous, namely the distribution of $(X_t - X_s)$ depends on $(t - s)$ only;
5. $\mathbb{R}_+ \colon t \mapsto X_t$ is continuous in probability, i.e.

$$\lim_{t\to s} \mathbb{P}(|X_s - X_t| > \epsilon) = 0, \quad \epsilon > 0.$$

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# 18. Distribution of a Lévy process

## Distribution

1. The distribution $\mu_1$ of $X_1$ is infinitely divisible, hence the characteristic function has the exponential form $\check{\mu}_1(\theta) = \mathrm{e}^{\psi(\theta)}$ with cumulant function given by th Lévy-Khinchin formula:

$$\psi(\theta) = i\gamma\theta - \frac{1}{2}\sigma^2\theta^2 + \int \left(\mathrm{e}^{i\theta y} - 1 - i\theta h(y)\right) \ \nu(dy).$$

2. The characteristic function of the distribution $\mu_t$ is $\check{\mu}_t(\theta) = \mathrm{e}^{t\psi(\theta)}$.
3. The finite-dimensional distributions of $X$ depend on $\mu_1$ only.

*Note:* Because of the independent increments and homogenuity, $\check{\mu}_s(\theta)\check{\mu}_t(\theta) = \check{\mu}_{s+t}(\theta)$, hence $\check{\mu}_t(\theta) = (\check{\mu}_1(\theta))^t$. Given $h$, the *generating triple* $(\gamma, \sigma^2, \nu)$ is unique.



# 19. Poisson random measure

## Definition ([Sato] Ch. 4. [Applebaum] Ch. 2)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(R, \mathcal{R}, \nu)$ be a $\sigma$-finite measure space.

- A random variable $N$ with values in $\overline{\mathbb{Z}}_+ = \{0, 1, 2, \dots, \infty\}$ is a (generalised) Poisson random variable with mean (intensity) $\lambda \in \overline{\mathbb{R}}_+ = [0, \infty]$ in cases: $\lambda = 0$, then $N = 0$ a.s.; $0 < \lambda < \infty$, $N \sim \mathrm{Poi}(\lambda)$; $\lambda = \infty$, $N = \infty$ a.s.
- A mapping $N \colon \mathcal{R} \times \Omega$ with values in $\overline{\mathbb{Z}}_+$ is a *Poisson random measure* if
  1. $N$ is a *transition measure*, i.e. for each fixed $\omega \in \Omega$, the mapping $A \mapsto N(A, \omega)$ is a measure;
  2. $\omega \mapsto N(A, \omega)$ is a Poisson random variable, denoted by $N(A)$, with intensity $\rho(A)$, $A \in \mathcal{R}$, where $\rho$ is a positive measure called the *intensity measure* of $N$;
  3. If $A_1, \dots, A_n \in \mathcal{R}$ are disjoint, then $N(A_1), \dots, N(A_n)$ are indepemdent.

# 20. Lévy-Itô decomposition

## Measure of the jumps

Let $X$ be a Lévy process.

- On the measurable space $(\mathbb{R}_+, \times\mathbb{R}_*, \mathcal{B}(\mathbb{R}_+, \times\mathbb{R}_*))$ consider for each $\omega \in \Omega$ the CÀDLÀG trajectory $t \mapsto X_t(\omega)$ and the countable set of jumps $J(\omega) = \{(t, \Delta X_t(\omega))\}$ of that trajectory.
- The measure $N(\cdot, \omega)\colon A \mapsto N(A, \omega) = \#\{(t, \Delta X_t(\omega)) \in A\}$, $A \in \mathcal{B}(\mathbb{R}_+, \times\mathbb{R}_*))$ is $\overline{\mathbb{Z}}_+$-valued and $\sigma$-finite, because it is finite on each set $[0, T[\times \{y||y| > \epsilon\}$, $T, \epsilon > 0$.

## The measure of the jumps is Poisson

Let $X$ be a Lévy process with generating triple $(\gamma, \sigma^2, \nu)$ and measure of jumps $N$. Let us define $\widetilde{\nu}(ds, dy) = ds\nu(dy)$. Then
$N$ is a Poisson measure with intensity measure $\widetilde{\nu}$.



# 21. Examples I

1. Let $W$ be a Wiener process. Then the generating triple is $\gamma = 0$, $\sigma^2 = 1$, $\nu = 0$, and there are no jumps, hence $N = 0$.
2. The process $X_t = \gamma t + \sigma W_t$, $\gamma \in \mathbb{R}$, $\sigma \geq 0$, is Lévy without jumps and generating triple $(\gamma, \sigma^2, 0)$.
3. Let $Z$ be a Poisson prosess with intensity $\lambda$ and jump times $T_1, T_2, \dots$. The generating triple is $\gamma = ih(1)$, $\sigma^2 = 0$, $\nu = \lambda\delta_1$. The measure $\widetilde{\nu}$ defines the integral $\int f(s, y)\, \widetilde{\nu}(ds, dy) = \lambda \int f(s, 1)\, ds$, in particular $\int_0^t \int_B f(y)\, \widetilde{(}ds, dy) = tf(1)(1 \in B)$. The set of jumps for the trajectory $X_\cdot(\omega)$ is $J(\omega) = \{(T_j(\omega), 1)|j \in \mathbb{N}\}$ and $N(A, \omega) = \#\{(T_j(\omega), 1)|(T_j(\omega), 1) \in A\}$. In particular, for $A = [0, t] \times B$, we have for $1 \notin B$ that $N([0, t] \times B, \omega) = \#\{(T_j(\omega), 1)|T_j \leq t, 1 \in B\} = 0$, othewise $N([0, t] \times B, \omega) = \#\{(T_j(\omega), 1)|T_j \leq t, 1 \in B\} = \#\{j \in \mathbb{N}|T_j \leq t\} = Z_t$, that is $N([0, t] \times B)$ is a Poisson random variable with intensity $\widetilde{\nu}([0, t] \times B) = t\nu(B)$. For each $\omega$ the

# 22. Examples II

trajectory $t \mapsto N([0,t] \times B, \omega) = N_t(B, \omega)$ is CÀDLÀG and $N(B)$ is a Poisson process, precisely $N(B) = 0$ if $1 \notin B$, otherwise $N(B) = Z$. This implies independence on disjoint intervals.

4. $X = W + Z$ is the prototype of the general case.
5. If $X$ has a compound Poisson triple, we obtain a non-trivial example of the contruction.

## Conclusion

Go to [Applebaum] Ch. 2 or [Sato] Ch. 4 if you want to read the end of the story.