3. INFINITELY DIVISIBLE DISTRIBUTIONS

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The following lecture notes are based on Sasvári [2] and Sato [3].

1. **Definition** (Convolution)**.** *Let X, Y be independent random variables with values in $\mathbb{R}^n$ and distributions μ_X, μ_Y, respectively. The* convolution $\mu_X * \mu_Y$ *is the distribution of $X+Y$, that is for all bounded $f \colon \mathbb{R}^n \to \mathbb{R}$*

$$\int f(z)\ (\mu_X * \mu_Y)(dz) = \mathrm{E}\left(f(X+Y)\right) = \iint f(x+y)\ \mu_X(dx)\mu_Y(dy).$$

2. **Proposition** (Convolution)**.**

(1) *If μ_1 has density p_1 with respect to the Lebesgue measure, then $\mu_1 * \mu_2$ has density $p_1 * \mu_2$ given by*

$$p_1 * \mu_2(z) = \int p_1(z-y)\ \mu_2(dy).$$

(2) *If both have densities p_1, p_2, respectively, with respect to the Lebesgue measure, then $\mu_1 * \mu_2$ has density $p_1 * p_2$ given by*

$$(p_1 * p_2)(z) = \int p_1(z-y)p_2(y)\ dy = \int p_1(x)p_2(z-x)\ dx.$$

(3) *If the measurable function $f \colon \mathbb{R}^n \to \mathbb{R}$, and μ is a measure, define $f * \mu(z) = \int f(z-x)\ \mu(dx)$ if the integral exists a.e, i.e. $\int |f(z-x)|\ \mu(dx) < \infty$ a.s in z. If the measure μ is finite and $f \in L^a(dx)$, $1 \leqslant a \leqslant \infty$, then then $f * \mu$ exists and $\|f * \mu\|_a \leqslant \|f\|_a$.*

Exercise.

(1) For each bounded $f \colon \mathbb{R}^n \to \mathbb{R}$

$$\int f(z)\ (\mu_1 * \mu_2)(dz) = \iint f(x+y)\ p_1(x)\ dx\ \mu_2(dy) =$$
$$\int \left(\int f(x+y)\ p_1(x)\ dy \right)\ \mu_2(dy) = \int \left(\int f(z)\ p_1(z-x)\ dz \right)\ \mu_2(dy) =$$
$$\int f(z) \left(\int\ p_1(z-y)\ \mu_2(dy) \right)\ dz.$$

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(2) For each bounded $f\colon \mathbb{R}^n \to \mathbb{R}$

$$\int f(z)\ (\mu_1 * \mu_2)(dz) = \iint f(x+y)\ p_1(x)dx\ p_2(y)dy =$$
$$\int \left(\int f(x+y)\ p_2(y)dy\right)\ p_1(x)dx = \int \left(\int f(z)\ p_2(z-x)dz\right)\ p_1(x)dx =$$
$$\int f(z) \left(\int p_1(x)p_2(z-x)dx\right)\ dz.$$

(3) From $\iint |f(z-x)|^a\ dx\mu(dz) = \int |f(y)|^a\, dy$.

□

3. **Exercise.**

(1) *Compute $\chi * f$ for $\chi = \delta_a$ and $f \in C_b$.*
(2) *Compute $\chi * f$ for $\chi = (\delta_a - \delta_b)^{*n}$, $n = 1, 2, 3$ and $f \in C_b$. If f is a polynomial of degree $n-1$, then $(\delta_a - \delta_b)^{*n} * f = 0$.*
(3) *Compute $\chi * f$ for $\chi(dx) = A^{-1}(0 \leqslant x \leqslant A)dx$. If χ is a complex measure, define $\widetilde{\chi}(B) = \overline{\chi(-B)}$. Compute $\chi * \widetilde{\chi}$ and $\chi * \widetilde{\chi} * f$, $f \in \mathbb{C}_b$.*
(4) *Let $mu_\sigma = \mathrm{N}(0, \sigma^2 I)$, and $f \in C_b$. Then $\lim_{\sigma \to 0} f * \mu_\sigma(z) = f(z)$.*

Solution.

(1) $\delta_a * f(y) = \int f(y-x)\ \delta_a(dx) = f(y-a)$ is the translation of f.
(2) $(\delta_a - \delta_b) * f(y) = f(y-a) - f(y-b)$. If f is constant, then $f(y-a) - f(y-b) = 0$. $(\delta_a - \delta_b)^{*2} * f(y) = f(y-2a) - 2f(y-a-b) + f(y-2b)$. If $f = u^t y + v$ is a polynomial of degree 1, then $f(y-a) - f(y-b) = u^t(y-a) - u^t(y-b) = u^t(b-a)$ is constant. In general, if f is a polynomial of degree n, then $(\delta_a - \delta_b) * f(y) = f(y-a) - f(y-b) = \sum_{k=1}^{x} \frac{1}{k!} f^{(k)}(y-b)(b-a)^k$ is a polynomial of degree at most $n-1$, e.g. $(y-a)^2 - (y-b)^2 = a(y-b)(b-a) + (a-b)^2$.
(3) Continuity and dominated convergence:

$$f * \mu_\sigma(y) = \int f(y-x)(2\pi)^{-n/2}\sigma^{-n}\mathrm{e}^{-\|x\|^2/2\sigma^2}\ dy =$$
$$\int f(y-\sigma z)(2\pi)^{-n/2}\mathrm{e}^{-\|x\|^2/2}\ dz \to \int f(y)(2\pi)^{-n/2}\mathrm{e}^{-\|x\|^2/2}\ dz = f(y).$$

□

4. **Definition** (Characteristic function, inverse Fourier tranform)**.**

(1) *Let X be a random variable in $\mathbb{R}^n$, with distribution μ_X. The* characteristic function *of X is the function $\phi_X\colon \mathbb{R}^n \to \mathbb{C}$ defined by*

$$\phi_X(t) = \mathrm{E}\left(\mathrm{e}^{i\langle t, X\rangle}\right) = \int \mathrm{e}^{i\langle t,x\rangle}\ \mu_(dx), \quad i = \sqrt{-1}.$$

(2) *The function $\check{\mu}_X(t) = \int \mathrm{e}^{i\langle t,x\rangle}\ \mu_(dx)$ is also called* inverse Fourier transform *of μ. If μ has a density f with respect to the Lebesgue measure, then the inverse Fourier transform of f is $\check{f}(t) = \int \mathrm{e}^{i\langle t,x\rangle} f(x)\ dx = \check{\mu}(t)$. If $f \in L^1(dx)$ then the Fourier transform $\hat{f}(t) = \int \mathrm{e}^{-i\langle t,x\rangle} f(x)\ dx$ is defined for all t and $\left\|\hat{f}\right\|_1 \leqslant \|f\|_1$.*

5. **Definition** (Positive definite)**.** *A function $f\colon \mathbb{R}^n \to \mathbb{C}$ is* positive definite *if for all $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}^n$, $c_1, \dots, c_m \in \mathbb{C}$ it holds $\sum_{i,j=1}^n c_i \overline{c_j} f(x_i - x_j) \geqslant 0$. In other words, for all $m \in \mathbb{N}$, $x_1, \dots, x_m \in \mathbb{R}^n$ the matrix $A = [f(x_i - x_j)]_{i,j=1}^m$ is positive definite, that is for all $u \in \mathbb{C}^m$, it holds $u^* A u \geqslant 0$, $u^* = \overline{u^t}$.*

6. **Definition** (Hermitian). *A function* $f\colon \mathbb{R}^n \to \mathbb{C}$ *is* Hermitian *if* $f(-t) = \overline{f(t)}$, $t \in \mathbb{R}^n$.

7. **Proposition** (Properties of positive definite functions).

(1) *If* f *is positive definite, then* f *is Hermitian and* $f(0) \geqslant 0$.

(2) *For a complex measure* μ, *define* $\widetilde{\mu}$ *by* $\int f(x)\, \widetilde{\mu}(dx) = \overline{\int f(-x)\, \mu(dx)}$. *Analogously, for each* $g\colon \mathbb{R}^n \to \mathbb{C}$, *define* $\widetilde{g}(t) = \overline{g(-t)}$. *If* f *is continuous and positive definite, then* $\mu * \widetilde{\mu} * f$ *is positive definite. Similarly for* $g * \widetilde{g} * f(t)$.

(3) *If* $g \in L^2(\mathbb{R}^n, \lambda; \mathbb{C})$, *that is* $\int |g(x)|^2 \ dx < \infty$, *then* $g * \widetilde{g}$ *exists and is positive definite.*

(4) *Let* X_t, $t \in \mathbb{R}^n$ *be a family of random variables such that* $\operatorname{Cov}(X_t, X_s) = \rho(t-s)$. *Then* ρ *is positive definite.*

(5) *If* f *is positive definite, then* $x \mapsto \mathrm{e}^{i\langle t,x\rangle} f(x)$ *is positive definite.*

(6) *If* f *is positive definite and integrable, the* $\int f(x)\ dx \geqslant 0$.

Exercise.

(1) Take $m = 1$ $c_1 = 1$, $x_1 = 0$: $c_1\overline{c_1}f(0-0) = f(0) \geqslant 0$. Take $m = 2$, $x_1 = 0$, $x_2 = x$: $|c_1|^2 f(0) + c_1\overline{c_2}f(-x) + c_2\overline{c_1}f(x) + |c_2|\, f(0) \geqslant 0$. In particular, with $c_1 = c_2 = 1$, we have $f(-x) + f(x) \in \mathbb{R}$, and, with $c_1 = 1$, $c_2 = i$, we have $i(f(x) - f(-x)) \in \mathbb{R}$. It follows $f(x) = [(f(-x)+f(x)) - i \cdot i(f(x) - f(-x))]/2$ and$f(-x) = [(f(-x)+f(x)) + i \cdot i(f(x) - f(-x))]/2$.

(2) It is enough to consider $\mu = \sum_{i=1}^m c_i \delta_{x_j}$, $\widetilde{\mu} = \sum_{j=1}^m \overline{c_j}\delta_{-x_j}$. In such a case

$$\mu * \widetilde{\mu} * f(z) = \iint f(z-x-y)\ \mu(dx)\widetilde{\mu}(dy) = \sum_{i,j=1}^m c_i\overline{c_j} f(z - x_i + x_j).$$

We check the positive definiteness with

$$\sum_{h,k=1}^M d_h\overline{d_k}\mu * \widetilde{\mu} * f(z) = \iint f(z-x-y)\ \mu(dx)\widetilde{\mu}(dy) =$$

$$\sum_{h,k=1}^M d_h\overline{d_k} \sum_{i,j=1}^m c_i\overline{c_j} f(z_h - z_k - x_i + x_j) = \sum_{h,k=1}^M \sum_{i,j=1}^m d_h\overline{d_k} c_i\overline{c_j} f((z_h - x_i) - (z_k - x_j)) =$$

$$\sum_{(h,i),(k,j)} (c_i d_h)\overline{(c_j d_k)}(f(y_{h,i} - y_{k,j}) \geqslant 0.$$

(3) The existence of $g * \widetilde{g}(y) = \int g(y-x)\overline{g(-x)}\ dx$ follows from

$$|g * \widetilde{g}(y)| \leqslant \sqrt{\int |g(y-x)|^2\ dx}\sqrt{\int \left|\overline{g(-x)}\right|^2\ dx} = \|g\|_2^2\,.$$

The positive definiteness is

$$\sum_{i,j=1}^m c_i\overline{c_j}\int g(y_i - y_j - x)\overline{g(-x)}\ dx = \int \sum_{i,j=1}^m c_i\overline{c_j} g(y_i - y_j - x)\overline{g(-x)}\ dx =$$

$$\int \sum_{i,j=1}^m c_i\overline{c_j} g(y_i - x)\overline{g(y_j - x)}\ dx = \int \left|\sum_{i=1}^m c_i g(y_i - x)\right|\ dx \geqslant 0.$$

(4) From the definition.

(5) From the definition.

(6) Let g_n be a a sequence of triangular functions such that $f(z)g_n(z) \to f(z)$. Write $g_n = h_n * \widetilde{h_n}$, h_n being uniform, and compute $\int f(x)\tau_n(x)\,dx$ as the value at 0 of a positive convolution. I.e. (n=1) let $h_n(x) = n^{-1/2}(0 \leqslant x \leqslant n)$ and define $g_n = h_n * \widetilde{h}_n$. Then $g_n(0) = 1$, $0 \leqslant g_n(x) \leqslant 1$ and $g_n(x) \to 1$, $n \to \infty$. From $\int f(x)g_n(x)\,dx = f * g_n * \widetilde{g}_n(0) \geqslant 0$ we obtain the result.

□

8. **Proposition.**

(1) *The characteristic function is uniformly continuous and* $\check{\mu}(0) = 1$.
(2) *If X is a random variable in $\mathbb{R}^n$ with characteristic function ϕ_X, for each $A \in \mathbb{R}^{m\times n}$ and $a \in \mathbb{R}^m$, the random variable $Y = a + AX$ has characteristic function $\phi_Y(s) = \mathrm{e}^{i\langle a,s\rangle}\phi_X(A^T s)$.*
(3) *If X_1 and X_2 are independent random variables with values in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, and characteristic functions ϕ_1, ϕ_2, respectively, then $X = (X_1, X_2)$ has characteristic function $\phi_X(t_1,t_2) = \phi_1(t_1)\phi_2(t_2)$.*
(4) *The characteristic function is Hermitian.*
(5) *The characteristic function is positive definite.*

Exercise.

(1) We have $\left|\mathrm{e}^{i\langle t+h,x\rangle} - \mathrm{e}^{i\langle t,x\rangle}\right| = \left|\mathrm{e}^{i\langle t,x\rangle}(\mathrm{e}^{i\langle h,x\rangle} - 1)\right| = \left|\mathrm{e}^{i\langle h,x\rangle} - 1\right| \leqslant 2$, and $\lim_{h\to 0} \mathrm{O}(h) = \int \left|\mathrm{e}^{i\langle h,x\rangle} - 1\right|\ \mu(dx) = 0$ by dominated convergence.
(2) From $\langle s, a + Ax\rangle = \langle s, a\rangle + \left\langle A^T s, x\right\rangle$.
(3) From independence and $\left\langle (t_1,t_2),(x_1,x_2)\right\rangle_{n_1+n_2} = \left\langle t_1, x_1\right\rangle_{n_1} + \left\langle t_2, x_2\right\rangle_{n_2}$.
(4) $\check{\mu}(-t) = \int \mathrm{e}^{i\langle -t,x\rangle}\ \mu(dx) = \int \overline{\mathrm{e}^{i\langle t,x\rangle}}\ \mu(dx) = \overline{\check{\mu}(t)}$.
(5)

$$\sum_{i,j=1}^n c_i\overline{c_j}\check{\mu}(t_1 - t_j) = \int \sum_{i,j=1}^n c_i\overline{c_j}\mathrm{e}^{i\langle t_i - t_j,x\rangle}\ \mu(dx) =$$

$$\int \sum_{i,j=1}^n c_i\overline{c_j}\mathrm{e}^{i\langle t_i,x\rangle}\overline{\mathrm{e}^{i\langle t_j,x\rangle}}\ \mu(dx) = \int \left|\sum_{i=1}^n c_i \mathrm{e}^{i\langle t_i,x\rangle}\right|^2\ \mu(dx) > 0$$

□

The following proposition requires the use of complex logarithms, which are not easily defined because the complex exponential function $\mathrm{e}^z = \mathrm{e}^{\Re z}\mathrm{e}^{i\Im z} = \mathrm{e}^{\Im z}(\cos(\Im z) + i\sin(\Im z))$ is not invertible as $\mathrm{e}^z = \mathrm{e}^{z+i2\pi k}$, $k \in \mathbb{Z}$.

9. **Proposition** (Cumulant function)**.** *Let $\phi\colon \mathbb{R}^n \to \mathbb{C}$ be the characteristic function of the probability measure μ and assume that $\phi(t) \neq 0$, $t \in \mathbb{R}^n$. There exists a unique continuous function $\psi\colon \mathbb{R}^n \to \mathbb{C}$ such that $\phi(t) = \mathrm{e}^{\psi(t)}$ and $\kappa(0) = 0$. Such a function is Hermitian. It is called the* cumulant function *of μ.*

Proof. In steps.

(1) The cumulant function is unique. In fact, if $\phi(t) = \mathrm{e}^{\psi_1(t)} = \mathrm{e}^{\psi_2(t)}$, then $\Re\psi_i(t) = \log\left(|\phi(t)|\right)$, $i = 1,2$, hence $\mathrm{e}^{i\Im\psi_1(t)} = \mathrm{e}^{i\Im\psi_2(t)}$, hence $\Im\psi_i(t) - \Im\psi_2(t) = 2\pi k(t)$. The function $t \mapsto k(t)$ is continuous and integer valued on $\mathbb{R}^n$, then constant and equal 0 at $t = 0$.
(2) The cumulant function is Hermitian. From $\phi(-t) = \overline{\phi(t)}$ it follows $\mathrm{e}^{i\Im\psi(-t)} = \mathrm{e}^{-i\Im\psi(t)}$, hence $\Im\psi(-t) + \Im\psi(t) = 2\pi k(t)$.

(3) Continuous argument. As ϕ is never zero, we can define define $f(t) = \phi(t)/|\phi(t)|$ so that $f\colon \mathbb{R}^n \to \mathbb{T} = \{z \in \mathbb{C} | |z| = 1\}$ is continuous and $f(0) = 1$. A *continuous argument* of f is a continuous $\theta\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(t) = \mathrm{e}^{i\theta(t)}$ and $\theta(0) = 0$. If a continuous argument exist, then $\phi(t) = |\phi(t)| \mathrm{e}^{i\theta(t)} = \mathrm{e}^{\log(|\phi(t)|)+i\theta(t)} = \mathrm{e}^{\psi(t)}$, with $\psi = \log|\phi| + i\theta$ continuous and $\psi(0) = \log 1 + \theta(0) = 0$.

(4) Consider a continuous $f\colon B \to \mathbb{T}$, $B \subset \mathbb{R}^n$, such that $f(B) \neq \mathbb{T}$. Choose $\mathrm{e}^{i\alpha} \in \mathbb{T} \backslash f(B)$. Then the function $z \mapsto \arg(\mathrm{e}^{-i\alpha} z) + \alpha$ is a continuous bijection of $\mathbb{T} \backslash \{\mathrm{e}^{i\alpha}\}$ onto $]\alpha, \alpha + 2\pi[$, hence $f(t) = \mathrm{e}^{i(\arg(\mathrm{e}^{-i\alpha} f(t))+\alpha)}$ so that $t \mapsto \theta(t) = \arg(\mathrm{e}^{-i\alpha} f(t)) + \alpha$ is a continuous argument of f on B.

(5) Let $f_1, f_2\colon B \to \mathbb{T}$, $B \subset \mathbb{R}^n$, be continuous. Assume f_1 has a continuous argument on B, $f_1 = \mathrm{e}^{i\theta_1}$, and $f_1(t) + f_2(t) \neq 0$, $t \in B$. Then $f_1/f_2\colon \mathbb{R}^n \to \mathbb{C}$ never equals -1, hence it has a continuous argument on B, $f_1/f_2 = \mathrm{e}^{i\theta}$, so that $f_2 = \mathrm{e}^{i(\theta_1 - \theta)}$.

(6) Consider the function $f^R\colon B(R) = \left\{t \middle| \|t\|^2 \leqslant R\right\} \times [0,1] \ni (t, \alpha) \mapsto f(\alpha t)$. The function f^R is continuous on a compact set, hence uniformly continuous, so that there exists a $n \in \mathbb{N}$ such that $\left|f^R(\alpha_1 t) - f^R(\alpha_2 t)\right| \leqslant 1$ for all t if $|\alpha_1 - \alpha_1| \leqslant 1/n$. Consider the sequence $f_j(t) = f^R\left(\frac{j}{n} t\right)$, $j = 0, 1, \ldots, n$. We proceed by finite induction on j. if $j = 0$ then $f_0(t) = f(0) = 1$ and the continuous argument is $\theta_0 = 0$. If there exist a continuous argument on $B(R)$ for f_j, $j < n$, as $|f_{j+1}(t) - f_j(t)| \leqslant 1$, the relation $f_{j+1}(t) + f_j(t) = 0$ is impossible, and the previous item shows that f_{j+1} has a continuous argument on $B(R)$. Finally, note that $f_n = f^R$.

(7) Each f has a continuous argument θ_R on $B(R)$ and $\theta_{R_1}(t) = \theta_{R_2}(t)$ for all $t \in B(\min(R_1, R_2))$ because $B(\min(R_1, R_2))$ is connected and the uniqueness argument applies. Then the global continuous argument is defined by its restrictions.

□

10. **Exercise.**

(1) *Let μ, ν, be probability measures on $\mathbb{R}^n$ with characteristic function respectively $\check{\mu}$, $\check{\nu}$. Then*

$$\int \mathrm{e}^{i\langle t,y\rangle} \check{\mu}(t)\ \nu(dt) = \int \check{\nu}(x+y)\ \mu(dx)$$

(2) *Let $X \sim \mu$ independent of $Y \sim \mathrm{N}(0,1)$ and $\nu \sim X + \sigma Y$. Let g_σ be the density of σY. Then $\nu = \mu * g_\sigma$ has density*

$$p_\sigma(t) = \int g_\sigma(t-x)\ \mu(dx) = (2\pi)^{-n/2} \sigma^n \int \check{g}_{1/\sigma}\ \mu(dx) = \int \mathrm{e}^{-i\langle x,y\rangle} \check{\mu}(x) g_{1/\sigma}(x)\ dx.$$

Proof.

(1) We have $\mathrm{e}^{i\langle t,y\rangle} \check{\mu}(t) = \mathrm{e}^{i\langle t,y\rangle} \int \mathrm{e}^{i\langle t,y\rangle} \mu(dy) = \int \mathrm{e}^{i\langle t,x+y\rangle}\ \mu(dy)$ and we can take the integral with respect to ν to get $\int \left(\mathrm{e}^{i\langle t,y\rangle} \check{\mu}(t)\right)\ \nu(dt) = \iint \mathrm{e}^{i\langle t,x+y\rangle}\ \mu(dy)\nu(dt) = \int \check{\nu}(x+y)\ \mu(dx)$.

(2) From the properties of the Gaussian density and the previous equality.

□

11. **Proposition** (Inversion theorems)**.** *Let denote by μ be a probability measure on $\mathbb{R}^n$ with inverse Fourier transform $\check{\mu}$.*

(1) *For all $f\colon \mathbb{R}^n \to \mathbb{R}$ continuous and with bounded support, $f \in C_{00}(\mathbb{R}^n)$,*

$$\int f(x)\ \mu(dx) = (2\pi)^{-n} \lim_{T\to\infty} \int_{-T}^{T} \cdots \int_{-T}^{T} \left(\int f(s) \mathrm{e}^{-i\langle s,t\rangle}\ ds\right) \check{\mu}(t)\ dt.$$

(2) *The mapping $\mu \mapsto \check{\mu}$ is 1-to-1.*
(3) *If $\check{\mu}$ is integrable, that is $\int |\check{\mu}(t)| \ dt < \infty$, then μ has a bounded and uniformly continuous density p with respect to the Lebesgue measure λ, and*

$$p(x) = (2\pi)^{-n} \int \check{\mu}(t) \mathrm{e}^{-i\langle i,x\rangle} \ dt = (2\pi)^{-n} \hat{\check{\mu}}(t).$$

Exercise.

(1) We have

$$\begin{aligned}
&\int_{-T}^{T} \cdots \int_{-T}^{T} \left(\int f(s) \mathrm{e}^{-i\langle s,t\rangle} \ ds \right) \check{\mu}(t) \ dt = \\
&\qquad \int_{-T}^{T} \cdots \int_{-T}^{T} \left(\int f(s) \mathrm{e}^{-i\langle s,t\rangle} \ ds \right) \left(\int \mathrm{e}^{i\langle t,y\rangle} \ \mu(dy) \right) \ dt = \\
&\qquad \int ds \ f(s) \int \mu(dy) \int_{-T}^{T} \cdots \int_{-T}^{T} dt \ \mathrm{e}^{i\langle t, y-s\rangle} = \\
&\qquad 2^n \int ds \ f(s) \int \mu(dy) \prod_{i=1}^{n} \frac{\sin(T(y_i - s_i))}{y_i - s_i} = \\
&\qquad 2^n \int \left(\int f(s) \prod_{i=1}^{n} \frac{\sin(T(y_i - s_i))}{y_i - s_i} \ ds \right) \ \mu(dy) = \qquad T(y-s) = u \\
&\qquad = 2^n \int f(y - T^{-1}u) \prod_{i=1}^{n} \frac{\sin(u_i)}{u_i} \ du \to \pi^n \quad T \to \infty
\end{aligned}$$

See another version in [4, §16.6].
(2) Follows from the previous inversion formula and the monotone class theorem. See also a direct proof based on Ex. . in [1, 14.1]
(3) Use the approximation with the Gaussian kernel of Ex. ., see [2, Th 1.3.6]

□

12. **Definition** (Weak convergence, convergence in distribution)**.**

(1) *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measure on $\mathbb{R}^n$* converges weakly *to a probability measure μ if for all bounded and continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \int f(x) \ \mu_n(dx) = \int f(x) \ \mu(dx), \quad f \in C_b(\mathbb{R}^n).$$

(2) *If $(X_n)_{n\in\mathbb{N}}$ and X are random variables in $\mathbb{R}^n$ we say that* $\lim_{n\to\infty} X_n = X$ in distribution *if* $\lim_{n\to\infty} \mathrm{E}\left(f(X_n)\right) = \mathrm{E}\left(f(X)\right)$, *$f \in C_b(\mathbb{R}^n)$.*

13. **Proposition.** *Let μ, ν, μ_n, $n \in \mathbb{N}$, be probability measures on $\mathbb{R}^n$. If $\lim_{n\to\infty} \mu_n = \mu$ weakly, then $\lim_{n\to\infty} \nu * \mu_n = \nu * \mu$ weakly.*

Proof. If $f \in C_b(\mathbb{R}^n)$, then for all x we have $(y \mapsto f(x+y) \in C_b(\mathbb{R}^n)$, and dominated convergence implies that

$$\lim_{n\to\infty} \int f(z)\ (\nu * \mu_n)(dz) = \lim_{n\to\infty} \iint f(x+y)\ \nu(dx)\mu_n(dy) =$$

$$\lim_{n\to\infty} \int \left(\int f(x+y)\ \mu_n(dy)\right)\ \nu(dx) = \int \left(\lim_{n\to\infty} \int f(x+y)\ \mu_n(dy)\right)\ \nu(dx) =$$

$$\int \left(\int f(x+y)\ \mu(dy)\right)\ \nu(dx) = \int f(z)\ (\nu * \mu)(dz).$$

□

14. **Proposition** (Lévy continuity theorem). *Let $(\phi_n)_{n\in\mathbb{N}}$ the the sequence of characteristic functions of the sequence of probability measures $(\mu_n)_{n\in\mathbb{Z}}$. If there exist the pointwise limit $\phi(t) = \lim_{n\to\infty} \phi_n(t)$, $t \in \mathbb{R}^n$, and the limit function ϕ is continuous at 0, then ϕ is a characteristic function of a probability measure μ. In such a case, for all bounded continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, we have $\lim_{n\to\infty} \int f\ d\mu_n = \int f\ d\mu$, that is the sequence $(\mu_n)_{n\in\mathbb{N}}$ weakly converges to μ.*

Proof. See [4, 18.1] or [1, Th. 19.1]. □

15. **Proposition.** *The mapping $\mu \mapsto \check{\mu}$ is 1-to-1 from probability measure to positive definite functions whose value is 1 at 0.*

Proof. □

16. **Proposition** (Bochner theorem). *If the function $\phi\colon \mathbb{R}^n \to \mathbb{C}$ is continuous, positive definite, and such that $\phi(0) = 1$, then there exists a probability measure μ such that $\phi = \check{\mu}$.*

Exercise. Assume first that ϕ is integrable. Let g_σ be the density of the Gaussian $\mathrm{N}(0, \sigma^2 I)$ with characteristic function $\check{g}_\sigma$,

$$g_\sigma(x) = (2\pi\sigma^2)^{-n/2} \mathrm{e}^{-\|x\|^2/2\sigma^2}, \quad \check{g}_\sigma(t) = \mathrm{e}^{-\sigma^2\|t\|^2/2}.$$

Note that $\check{g}_\sigma(t) = (2\pi)^{-n/2} g_{1/sigma}(t)$. Let us compute

$$\begin{aligned}
\phi * \check{g}_\sigma(t) &= \int \phi(s)\check{g}_\sigma(t-s)\ ds \\
&= \int \phi(s) \left(\int \mathrm{e}^{i\langle t-s,x\rangle} g_\sigma(x)\ dx\right)\ ds \\
&= \int \mathrm{e}^{i\langle t,x\rangle} g_\sigma(x) \left(\int \mathrm{e}^{-i\langle s,x\rangle}\phi(s)\right)\ dx \\
&= \int \mathrm{e}^{i\langle t,x\rangle} g_\sigma(x)\hat{\phi}(x)\ dx
\end{aligned}$$

As $\hat{\phi}$ is nonnegative being the Fourier transform of a positive definite function, and $\hat{\phi} \leqslant \int |\phi|$, we can renormalize $g_\sigma \cdot \hat{\phi}$ to get a probability density, so that

$$\frac{\phi * \check{g}_\sigma(t)}{\phi * \check{g}_\sigma(0)} = \frac{\phi * g_{1/\sigma(t)}}{\phi * g_{1/\sigma}(0)}$$

is a characteristic function. As $\sigma \to 0$, letting $\sigma s = u$,

$$\phi * \check{g}_\sigma(t) = \int \phi(t-s)\mathrm{e}^{-\sigma^2\|s\|^2/2} ds = \sigma^{-1} \int \phi(t - \sigma^{-1}u)\mathrm{e}^{-\|u\|^2/2} du,$$

$$\frac{\phi * \check{g}_\sigma(t)}{\phi * \check{g}_\sigma(0)} = \frac{\int \phi(t - \sigma^{-1}) \mathrm{e}^{-\|u\|^2/2} du}{\int \phi(-\sigma^{-1}u) \mathrm{e}^{-\|u\|^2/2} du} \to \frac{(2\pi)^{-n/2} f(t)}{(2\pi)^{-n/2} f(0)} = f(t).$$

If ϕ is not integrable, for each n the function $\phi_n \colon t \mapsto \phi(t)\check{g}_{1/n}(t)$ are positive definite, integrable, 1 at 0, and $\lim_{n\to\infty} \phi_n(t) = \phi(t)$. □

17. **Definition** (Infinite divisibility).

- *A random variable X is* infinitely divisible *if for all $n \in \mathbb{N}$ there exist IID random variables $X_1, \dots, X_n$ such that $X \sim X_1 + \cdots + X_n$.*
- *Equivalently, a probability measure μ is* infinitely divisible *if for all $n \in \mathbb{N}$ there exists a probability measure μ_n such that $\mu = (\mu_n)^{*n}$.*
- *Equivalently, a characteristic function ϕ is* infinitely divisible *if for all $n \in \mathbb{N}$ there exists a characteristic function ϕ_n such that $\phi = (\phi_n)^n$.*

18. **Proposition.**

(1) *If the characteristic functions ϕ, ϕ_1 are infinitely divisible, then $\overline{\phi}$, $|\phi|^2$, $\phi\phi_1$ are infinitely divisible.*
(2) *Each infinite divisible characteristic function ϕ has a cumulant function, $\phi = \mathrm{e}^{\psi}$.*

Exercise.

(1) If ϕ and ϕ_1 are the characteristic functions of the random variables X and X_1, respectively, then $\overline{\phi}$ is the characteristic function of $-X$, $|\phi|^2$ of $X - X'$, X' being an independent copy of X, $\phi\phi_1$ of $X + X_1$, X and X_1 independent.
(2) Because of Proposition 9 we want $\phi(t) \neq 0$, $t \in \mathbb{R}^n$. For all $n \in \mathbb{N}$, let $\phi = (\phi_n)^n$. Then $|\phi|^2 = |\phi_n|^{2n}$, hence $|\phi|^{2/n} = |\phi_n|^2$ is a characteristic function for all n. The limit $\phi_*(t) = \lim_{n\to\infty} |f|^{2/n}$ is $\phi_*(t) = 1$ if $\phi(t) \neq 0$ and $\phi_*(t) = 0$ if $\phi(t) = 0$. As $\phi(t) =\neq 0$ in a neighborhood of 0, then ϕ_* is a characteristic function equal to 1 in a neighborhood of 0, hence it is a characteristic function, hence continuous, so that the case $\phi(t) = 0$ is impossible.

□

19. **Exercise** (Table of infinitely divisible characteristic functions).

Sampling. If ϕ_i, $i = 1, \dots, n$, are infinitely divisible, then $\phi_1 \otimes \cdots \otimes \phi_n$ is infinitely divisible.

Affine transformation. If ϕ is infinitely divisible, then $s \mapsto \mathrm{e}^{i\langle s,\mu\rangle}\phi(A^T s)$ is infinitely divisible. If ψ is the cumulant function, the transformed cumulant is $s \mapsto i\langle s, \mu\rangle + \psi(A^t s)$.

Dirac. The Dirac distribution δ_μ has characteristic function $\phi(t) = \mathrm{e}^{i\langle t,\mu\rangle}$ and cumulant function $\psi(t) = i\langle t, \mu\rangle$.

Poisson. If $X \sim \text{Poisson}(\lambda)$, then X has characteristic function $\phi(t) = \mathrm{e}^{\lambda(\mathrm{e}^{it}-1)}$. It is infinitely divisible. The cumulant function is

$$\psi(t) = \lambda(\mathrm{e}^{it} - 1) = \int (\mathrm{e}^{ity} - 1)\ (\lambda\delta_1)(dy) = \int (\mathrm{e}^{ity} - 1)\ \nu(dy)$$

Gaussian. If $X \sim \text{Normal}_n(0, I)$, then X has characteristic function $\phi(t) = \mathrm{e}^{-\|t\|/2}$. It is infinitely divisible. The cumulant function is $\psi(t) = -\frac{1}{2}\|t\|$.

Gamma. If $X \sim \text{Gamma}(\gamma, \lambda)$, then X has characteristic function $\phi(t) = (1-i\lambda^{-1}t)^{-\gamma}$. It is infinitely divisible with n-th root $\phi_n(t) = (1-i\lambda^{-1}t)^{-\gamma/n}$ and $\psi(t) = \lim_{n\to\infty} n(\phi_n(t)-1)$. Let us consider the measure

$$\nu(dy) = (\gamma y)^{-1} \mathrm{e}^{-\lambda y} (y > 0)\ dy.$$

We have

$$\begin{aligned}
\int (\mathrm{e}^{ity} - 1)\ \nu(dy) &= \gamma^{-1} \int_0^\infty \frac{\mathrm{e}^{ity} - 1}{y} \mathrm{e}^{-\lambda y}\ dy \\
&= i\gamma^{-1} \int_0^\infty \frac{\mathrm{e}^{ity} - 1}{iy} \mathrm{e}^{-\lambda y}\ dy \\
&= i\gamma^{-1} \int_0^\infty \left(\int_0^t \mathrm{e}^{isy}\ ds \right) \mathrm{e}^{-\lambda y}\ dy \\
&= i\gamma^{-1} \int_0^t ds \int_0^\infty dy\ \mathrm{e}^{-(\lambda - is)y} \\
&= i\gamma^{-1} \int_0^t (\lambda - is)^{-1}\ ds \\
&= -\gamma^{-1} \log(\lambda - is) \Big|_{s=0}^{s=t} \\
&= \log\left((1 - i\lambda^{-1}t)^{-\gamma} \right).
\end{aligned}$$

Hence

$$\psi(t) = \int (\mathrm{e}^{ity} - 1)\ \nu(dy).$$

Note that the measure μ is infinite, but $y \mapsto \mathrm{e}^{ity} - 1$ is integrable, because

$$\left|\mathrm{e}^{ity} - 1\right| = \left| i \int_0^{ty} \mathrm{e}^{iu}\ du \right| \leqslant |ty|,$$

so that

$$\int \left|\mathrm{e}^{ity} - 1\right|\ \nu(dy) \leqslant \gamma^{-1} |t| \int_0^\infty \mathrm{e}^{-\lambda y}\ dy = |t| / \gamma\lambda.$$

An other integral form is of interest. Let $h \colon \mathbb{R} \to \mathbb{R}$ be bounded, continuous, asymmetric $h(-y) = -h(y)$, equal to the identity $h(y) = y$ if $|y| < 1$ and constant for $|y| \geqslant 1$. Then

$$\int ith(y)\ \nu(dy) = it\gamma^{-1} \int_0^1 \mathrm{e}^{-\lambda y}\ dy + it\gamma^{-1} \int_1^\infty y^{-1} \mathrm{e}^{-\lambda y}\ dy = \\
it \left(\lambda^{-1}(\mathrm{e}^{-\lambda} - 1) + E_1(\lambda) \right) = it\mu$$

so that

$$\psi(t) = it\mu + \int (\mathrm{e}^{ity} - 1 - ith(y))\ \nu(dy).$$

The integrand in this form is a function that equals the Taylor remainder of order 2. In fact, if $|y| < 1$, then

$$\left|\mathrm{e}^{ity} - 1 - ith(y)\right| = \left|\mathrm{e}^{ity} - 1 - ity\right| \leqslant \frac{t^2y^2}{2} < \frac{t^2}{2}.$$

With the same function as above,

$$\int ith(y)\ \nu(dy) = it\lambda \int_{|y|<1} y\ \sigma(dy) + it\lambda \int_{|y|\geqslant 1} \text{sign}(y)\ \sigma(dy) = it\mu,$$

so that

$$\psi(t) = it\mu + \int (\mathrm{e}^{ity} - 1 - ith(y))\ \nu(dy).$$

Compound Poisson. Let N be a Poisson process with intensity λ and $(X_n)_{n\in\mathbb{N}}$ be IID with distribution σ. Assume N and $(X_n)_{n\in\mathbb{N}}$ independent. The process defined by $Y_t = \sum_{k=1}^{N_t} X_k$ is called *compound Poisson.* The characteristic function of Y_1 is

$$\phi(t) = \mathrm{E}\left(\mathrm{e}^{it\left(\sum_{k=1}^{N_1} X_k\right)}\right) = \sum_{n=0}^{\infty} \mathrm{E}\left(\mathrm{e}^{it\left(\sum_{k=1}^{n} X_k\right)}\ (N_1 = n)\right) = \sum_{n=0}^{\infty} (\check{\sigma}(t))^n \frac{\lambda^n}{n!}\mathrm{e}^{-\lambda} = \mathrm{e}^{\lambda((\check{\sigma}(t))^n - 1)}.$$

The cumulant function is

$$\psi(t) = \lambda((\check{\sigma}(t))^n - 1) = \int (\mathrm{e}^{ity} - 1)\ (\lambda\sigma)(dy) = \int (\mathrm{e}^{ity} - 1)\ \nu(dy).$$

Approximation by Compound Poisson. Let g_1 be the Gaussian density $\mathrm{N}(0,1)$. Then $\check{g}_1(x) = \mathrm{e}^{-\|x\|^2/2}$ with cumulant function $\psi(x) = -\frac{1}{2}\|x\|^2$. Consider the infinite divisibility, $\frac{1}{m}\psi(x) = \frac{1}{2m}\|x\|^2$, with is the cumulant function of $\mathrm{e}^{-\|x\|^2/2m} = \check{g}_{m^{-1/2}}(x)$, i.e. $\mathrm{N}(0, m^{-1})$. Consider the distribution Compound Poisson $\mathrm{CP}(m, g_{m^{-1/2}})$, i.e. the distribution of $\sum_{k=1}^{N} X_i$, with $N \sim \mathrm{Poi}(m)$, $(X_k)_k$ IID $\mathrm{N}(0, m^{-1})$. The characteristic function is

$$t \mapsto \mathrm{e}^{m(\check{g}_{m^{-1/2}}(t) - 1)} = \mathrm{e}^{m(\mathrm{e}^{-m^{-1}\|x\|^2/2} - 1)},$$

and cumulant function

$$t \mapsto m(\mathrm{e}^{-m^{-1}\|x\|^2/2} - 1) = \int (\mathrm{e}^{i\langle t,x\rangle} - 1)\ (mg_{m^{-1/2}})(x)dx.$$

The left hand side converges to $-\|x\|^2/2$. Note the peculiar convergence of the right hand side, were $g_{m^{-1/2}}(x)dx$ converges weakly toδ_0, while $mg_{m^{-1/2}}(x)dx$ does not converge weakly, as for all f integrable

$$\int f(x) m g_{m^{-1/2}}(x)\ dx = \int m f(m^{-1/2}x)\ g_1(x)\ dx.$$

20. **Definition.** *A function $f\colon \mathbb{R}^n \to \mathbb{C}$ is* conditionally positive definite *if it is Hermitian and $\sum_{i,j=1}^{m} c_i\overline{c_j} f(t_i - t_j) \geqslant 0$ for all $t_1, \dots, t_m \in \mathbb{R}^n$, $c_1, \dots, c_n \in \mathbb{C}$, $\sum_{i=1}^{m} c_i = 0$.*

21. **Proposition** (Lévy-Kinchin formula)**.** *Let the μ be a probability measure on $\mathbb{R}^n$ with characteristic function $\check{\mu}$ and cumulant function ψ, $\check{\mu}(t) = \mathrm{e}^{\psi(t)}$. The following conditions are equivalent:*

(1) *The probability measure is infinitely divisible.*
(2) *For all $n \in \mathbb{N}$ the function $t \mapsto \mathrm{e}^{\frac{1}{n}\psi(t)}$ is positive definite.*
(3) *The cumulant function is conditionally positive definite.*
(4) *The cumulant function has the following form*

$$\psi(t) = i\langle t, \mu\rangle - \frac{1}{2}\langle \Gamma t, t\rangle + \int \left(\mathrm{e}^{i\langle t,y\rangle} - 1 - i\langle t, h(y)\rangle\right)\ \nu(dy), \tag{1}$$

where

(a) *$h\colon \mathbb{R}^n \to \mathbb{R}^n$ is bounded, continuous, antisymmetric $h(-t) = -h(t)$, equal to the identity $h(t) = t$ in a neighborhood of 0*[1]*;*
(b) *$a, \mu \in \mathbb{R}$;*
(c) *Γ is a symmetric and positive definite $n \times n$ matrix;*

[1]Other choices are possible, cf [2], [3].

(d) ν *is a positive measure on* $\mathbb{R}^n$ *such that* $\nu(\{0\}) = 0$, *for all* $a > 0$, $\nu\{t | \|t\| > a\} < \infty$, *and* $\int \|y\|^2 \ \nu(dy) < \infty$.

Given h, the decomposition is unique and it is called Lévy-Khinchin formula. *The triple* (μ, Γ, ν) *is called the* Lévy triple *and* ν *is called the* Lévy measure.

Elements of the proof.

$(1 \Rightarrow 4)$ As ϕ is infinitely divisible, there exists characteristic functions ϕ_n such that $\phi(t) = (\phi_n(t))^n$, $n = 1, 2, \dots$. From the existence and uniqueness of the cumulant function, $\phi = \mathrm{e}^{\psi}$, as $\phi(t) = (\phi_n(t))^n = (\mathrm{e}^{\phi_n})^n = \mathrm{e}^{n\psi_n(t)}$, we obtain $\psi_n = \frac{1}{n}\psi$. The function $t \mapsto \mathrm{e}^{n(\phi_n(t)-1)}$ is the characteristic function of the compound Poisson $\mathrm{CP}(n, \mu_n)$, $\check{\mu}_n = \phi_n$, that is the characteristic function of $\sum_{k=0}^{N_n} X_k$, with $N_n \sim \mathrm{Poi}(n)$, $X_0 = 0$, $X_1, X_2, \dots$ IID μ_n, $N_n, X_1, X_2, \dots$ independent. As

$$\mathrm{e}^{n(\phi_n(t)-1)} = \exp\left(n(\mathrm{e}^{\frac{1}{n}\psi(t)} - 1)\right) = \exp\left(\frac{\mathrm{e}^{\frac{1}{n}\psi(t)} - 1}{1/n}\right) \to \mathrm{e}^{\psi(t)} = \phi(t), \quad n \to \infty,$$

we have obtained that every infinitely divisible distribution is the limit of compound Poisson distribution, which are themselves infinitely divisible.

The cumulant function of the compound Poisson is

$$\psi_n^{(c)}(t) = \int (\mathrm{e}^{ity} - 1) \ \nu_n(dy) = i\langle t, \mu_n\rangle + \int (\mathrm{e}^{ity} - 1 - i\langle t, h(y)\rangle) \ \nu_n(dy),$$

with $\nu_n(dy) = n\mu_n(dy)$ and $\mu_n = \int h(y) \ \nu_n(dy)$. Note that the Gaussian term is missing. It is possibly produced in the limit as $n \to \infty$.

$(4 \Rightarrow 3)$ The function $g(t) = i\langle t, \mu\rangle - \frac{1}{2}\langle \Gamma t, t\rangle$ is such that for all $\alpha > 0$ the function $f(t) = \mathrm{e}^{\alpha g(t)}$ is the characteristic function of $\mathrm{N}(\alpha\mu, \alpha\Gamma)$, hence positive definite. It follows that $(\mathrm{e}^{\alpha g(t)} - 1)/\alpha$ is conditionally positive definite and so is $g(t) = \lim_{\alpha \downarrow 0}(\mathrm{e}^{\alpha g(t)} - 1)/\alpha$. A direct computation shows that the integral part of (1) is conditionally positive definite.

$(3 \Rightarrow 2)$ The function $f(t) = \frac{1}{n}\psi(t)$ is conditionally positive definite for all n and $f(0) = 0$. Let us show that for all $t_1, \dots, t_m \in \mathbb{R}$ the matrix $[f(t_i - t_j) - f(t_i) - \overline{f(t_j)}]_{i.j=1}^m = [f(t_i - t_j) - f(t_i) - f(-t_j)]_{i.j=1}^m$ is positive definite. Chose $c_1, \dots, c_m \in \mathbb{C}$, $t_{m+1} = 0$, and c_{m+1} such that $\sum_{i=1}^{m+1} c_i = 0$. We have

$$\begin{aligned}
&\sum_{i.j=1}^m c_i\overline{c_j}(f(t_i - t_j) - f(t_i) - \overline{f(t_j)}) = \\
&\qquad \sum_{i.j=1}^m c_i\overline{c_j} f(t_i - t_j) - \sum_{i.j=1}^m c_i\overline{c_j} f(t_i) - \sum_{i.j=1}^m c_i\overline{c_j} f(-t_j) = \\
&\qquad \sum_{i.j=1}^m c_i\overline{c_j} f(t_i - t_j) + \overline{c_{m+1}} \sum_{i=1}^m c_i f(t_i) + c_{m+1} \sum_{j=1}^m \overline{c_j} f(-t_j) = \\
&\qquad\qquad \sum_{i,j=1}^{m+1} c_i\overline{c_j} f(t_i - t_j) \geqslant 0.
\end{aligned}$$

The matrix $[\mathrm{e}^{a_{ij}}]$ is positive definite if the matrix $[a_{ij}]$ is positive definite, then the matrix $[\mathrm{e}^{f(t_i - t_j) - f(t_i) - \overline{f(t_j)}}]_{i.j=1}^m$ s positive definite and also

$$\sum_{i,j=1}^m c_i\overline{c_j}\mathrm{e}^{f(t_i - t_j)} = \sum_{i,j=1}^m (c_i\mathrm{e}^{f(t_i)})\overline{c_j\mathrm{e}^{f(t_j)}}\mathrm{e}^{f(t_i - t_j) - f(t_i) - \overline{f(t_j)}} \geqslant o$$

which shows that the function $t \mapsto e^f(t) = e^{\frac{1}{n}\psi(t)}$ is positive definite.

(2 ⇒ 1) The characteristic function $\phi(t) = e^{\psi(t)}$ is infinitely divisible because for all n $\phi(t) = (e^{\frac{1}{n}\psi(t)})^n$.

Uniqueness Assume there are two representations, so that

$$i\langle t,\mu\rangle_1 - \frac{1}{2}\langle \Gamma_1 t,t\rangle + \int \left(e^{i\langle t,y\rangle} - 1 - i\langle t,h(y)\rangle\right)\ \nu_1(dy) = i\langle t,\mu\rangle_2 - \frac{1}{2}\langle \Gamma_2 t,t\rangle + \int \left(e^{i\langle t,y\rangle} - 1 - i\langle t,h(y)\rangle\right)\ \nu_2(dy),$$

hence

$$f(t) = \int \left(e^{i\langle t,y\rangle} - 1 - i\langle t,h(y)\rangle\right)\ \nu_1(dy) - \int \left(e^{i\langle t,y\rangle} - 1 - i\langle t,h(y)\rangle\right)\ \nu_2(dy),$$

where $f(t)$ is a polynomial of degree at most 2. The convolution of of this equality with $\chi = (\delta_0 - \delta_x)^{*3}$ gives

$$\int \hat{\chi}(y) e^{i\langle x,y\rangle}\ \nu_1(dy) = \int \hat{\chi}(y) e^{i\langle x,y\rangle}\ \nu_2(dy),$$

that is $(1 - e^{i\langle x,y\rangle})^3 \nu_1(dy) = (1 - e^{i\langle x,y\rangle})^3 \nu_2(dy)$, which implies $\nu_1 = \nu_2$. If the integrals are equal, then the first parts are equal.

□

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