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κ-exponential models from the geometrical viewpoint

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Abstract. We discuss the use of Kaniadakis' κ-exponential in the construction of a statistical manifold modelled on Lebesgue spaces of real random variables. Some algebraic features of the deformed exponential models are considered. A chart is defined for each strictly positive densities; every other strictly positive density in a suitable neighborhood of the reference probability is represented by the centered $\ln_\kappa$ likelihood.

1 Introduction

G. Kaniadakis [1–4], based on arguments from Statistical Physics and Special Relativity, has defined the *κ-deformed exponential* for each $x \in \mathbb{R}$ and $-1 < \kappa < 1$ to be

$$\exp_\kappa(x) = \exp\left(\int_0^x \frac{dt}{\sqrt{1+\kappa^2 t^2}}\right), \tag{1}$$

with special cases

$$\exp_\kappa(x) = \begin{cases} \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{\frac{1}{\kappa}}, & \text{if } \kappa \neq 0, \\ \exp x, & \text{if } \kappa = 0, \end{cases} \tag{2}$$

and derivation formulæ

$$\exp_\kappa{}'(x) = (1+\kappa^2 x^2)^{-1/2} \exp_\kappa(x) > 0 \tag{3}$$

$$\exp_\kappa{}''(x) = \frac{\sqrt{1+\kappa^2x^2} - \kappa^2 x}{1+\kappa^2 x^2} \exp_\kappa{}'(x) > 0 \tag{4}$$

For each $\kappa \neq 0$, the indeterminate $y = (\exp_\kappa(x))^\kappa$ and x are related by the polynomial equation

$$y^2 - 2\kappa x y - 1 = 0 \tag{5}$$

Therefore, the graph of $(\exp_\kappa)^\kappa$ is the upper branch of a hyperbola:

$$x = \frac{1}{2\kappa}\left(y - \frac{1}{y}\right), \quad y > 0. \tag{6}$$

For each given κ, the function $\exp_\kappa$ maps $\mathbb{R}$ unto $\mathbb{R}_>$, it is strictly increasing and it is strictly convex. If $\kappa \neq 0$, its inverse function is

$$\ln_\kappa(y) = \frac{y^\kappa - y^{-\kappa}}{2\kappa}, \quad y > 0, \tag{7}$$

with derivative

$$\ln_\kappa{}'(y) = \frac{y^\kappa + y^{-\kappa}}{2}\frac{1}{y}, \tag{8}$$

and it is called *κ-deformed logarithm.* The function $\ln_\kappa$ maps $\mathbb{R}_>$ unto $\mathbb{R}$, is strictly increasing and is strictly concave. Both deformed exponential and logarithm functions $\exp_\kappa$ and $\ln_\kappa$ reduce to the ordinary exp and ln functions when $\kappa \to 0$. Moreover, the algebraic properties of the exponential and logarithmic function are partially preserved, because

$$\exp_\kappa(x)\exp_\kappa(-x) = 1, \quad \ln_\kappa(y) + \ln_\kappa\left(y^{-1}\right) = 0. \tag{9}$$

Commutative group operations $(\mathbb{R}, \overset{\kappa}{\oplus})$ and $(\mathbb{R}_>, \overset{\kappa}{\otimes})$ are defined in such a way that $\exp_\kappa$ is a group isomorphism from $(\mathbb{R}, +)$ onto $(\mathbb{R}_>, \overset{\kappa}{\otimes})$ and also from $(\mathbb{R}, \overset{\kappa}{\oplus})$ onto $(\mathbb{R}_>, \times)$:

$$\exp_\kappa\left(x_1 \overset{\kappa}{\oplus} x_2\right) = \exp_\kappa(x_1)\exp_\kappa(x_2), \tag{10}$$

$$\exp_\kappa(x_1 + x_2) = \exp_\kappa(x_1) \overset{\kappa}{\otimes} \exp_\kappa(x_2), \tag{11}$$

and, equivalently,

$$\ln_\kappa\left(y_1 \overset{\kappa}{\otimes} y_2\right) = \ln_\kappa(y_1) + \ln_\kappa(y_2), \tag{12}$$

$$\ln_\kappa(y_1 y_2) = \ln_\kappa(y_1) \overset{\kappa}{\oplus} \ln_\kappa(y_2). \tag{13}$$

Therefore, the binary operations $\overset{\kappa}{\oplus}$ and $\overset{\kappa}{\otimes}$ are defined by

$$x_1 \overset{\kappa}{\oplus} x_2 = \ln_\kappa(\exp_\kappa(x_1)\exp_\kappa(x_2)) \tag{14}$$

$$y_1 \overset{\kappa}{\otimes} y_2 = \exp_\kappa(\ln_\kappa(y_1) + \ln_\kappa(y_2)) \tag{15}$$

Because of (9), the deformed operation have the same inverse that the usual operations:

$$x_1 \overset{\kappa}{\oplus} (-x_2) = 0, \quad y_1 \overset{\kappa}{\otimes} y_2^{-1} = 1. \tag{16}$$

The operation $\overset{\kappa}{\otimes}$ is defined on positive real numbers. However, (15) can be extended by continuity to non-negative real numbers:

$$0 \overset{\kappa}{\otimes} y_2 = y_1 \overset{\kappa}{\otimes} 0 = 0 \overset{\kappa}{\otimes} 0 = 0 \tag{17}$$

We want to derive defining relations for the κ-deformed operations in polynomial form. This is obtained by repeated use of (5) followed by algebraic elimination of the unwanted indeterminate. Symbolic computations have been done with [5]. First, we want to find $x = x_1 \overset{\kappa}{\oplus} x_2$, i.e. such that $\exp_\kappa(x) = \exp_\kappa(x_1)\exp_\kappa(x_2)$. Let $y_1 = (\exp_\kappa(x_1))^\kappa$ and $y_2 = (\exp_\kappa(x_2))^\kappa$. As

$$(\exp_\kappa(x))^\kappa = (\exp_\kappa(x_1)\exp_\kappa(x_2))^\kappa = y_1 y_2, \tag{18}$$

Equation (5) gives a system of three quadratic equations in the indeterminates $x_1, x_2, y_1, y_2, x, \kappa$. Algebraic elimination of y_1, y_2 gives the polynomial equation

$$x^4 - 2\left(2\kappa^2 x_1^2 x_2^2 + x_1^2 + x_2^2\right) x^2 + \left(x_1^2 - x_2^2\right)^2 = 0, \tag{19}$$

whose solution gives

$$x_1 \overset{\kappa}{\oplus} x_2 = x_1\sqrt{1+\kappa^2 x_2^2} + x_2\sqrt{1+\kappa^2 x_1^2}. \tag{20}$$

We will use later on the derivation formula

$$\frac{\partial(x_1 \overset{\kappa}{\oplus} x_2)}{\partial x_1} = \sqrt{1+\kappa^2 x_2^2} - \frac{\kappa^2 x_1 x_2}{\sqrt{1+\kappa^2 x_1^2}}. \tag{21}$$

Second, we want to find $z = \left(y_1 \overset{\kappa}{\otimes} y_2\right)^\kappa$. As before, $y_1 = (\exp_\kappa(x_1))^\kappa$, $y_2 = (\exp_\kappa(x_2))^\kappa$, while

$$z = (\exp_\kappa(x_1 + x_2))^\kappa. \tag{22}$$

Equation (5) gives three quadratic equations in the indeterminates $x_1, x_2, y_1, y_2, z, \kappa$. Elimination of x_1, x_2 gives the polynomial equation

$$y_1 y_2 z^2 + (1 - y_1 y_2)(y_1 + y_2) z - y_1 y_2 = 0 \tag{23}$$

It is remarkable that this equation does not depend on κ. An explicit solution is obtained by solving the quadratic equation. A possibly more suggestive solution is obtained as follows. First, we reduce to the monic equation

$$z^2 + \left(1 - \frac{1}{y_1 y_2}\right)(y_1 + y_2) z - 1 = 0 \tag{24}$$

and denote the two solutions as $z > 0$ and $-1/z$. Therefore,

$$z - \frac{1}{z} = \left(y_1 - \frac{1}{y_1}\right) + \left(y_2 - \frac{1}{y_2}\right). \tag{25}$$

The κ-logarithm defined in (7) is strictly related to a family of transformation which is well known in Statistics under the name of Box-Cox transformation [6] or power transform. For data vector $y_1, \ldots, y_n$ in which each $y_i > 0$, the power transform is:

$$y_i^{(\lambda)} \propto \frac{y_i^\lambda - 1}{\lambda} \tag{26}$$

The parameter λ is estimated in order to get the best fit to the Normal distribution of the transformed data vector. When compared with the Box-Cox transformation, the κ-deformed logarithm $x = \ln_\kappa(y)$ has the extra feature of the symmetry induced by the term $-y^{-\kappa}$ and it would be interesting to study it from the point of view of transformations to normality. We are not further discussing this issue here.

J. Naudts [7–10] has presented a general discussion of a class of deformed exponentials, including $\ln_\kappa$ and $\exp_\kappa$, in Information Theory and Statistical Physics, via a notion of generalised entropy. We are not discussing generalised entropies in this paper. The purpose of the present paper is to extend to κ-deformations the non-parametric and geometric approach to statistical manifolds as it was developed by the Author and co-workers in [11–15]. Such an approach is designed to present in a non-parametric and fully geometric way the S-i Amari Information Geometry, see e.g. [16] and the monograph by Amari and Nagaoka [17]. According to that references, the Box-Cox transformations are applied to probability densities and re-named α-embeddings.

Let Ω be any set, $\mathcal{F}$ a σ-algebra of subsets, μ a reference probability measure, e.g. the uniform distribution. A density p of the measure space $(\Omega, \mathcal{F}, \mu)$ is a non-negative random variable such that $\int p\, d\mu = 1$. We write the expected value of a random variable U with respect to the probability measure $p \cdot \mu$ as $\mathrm{E}_p[U] = \int U p\, \mu$. For $a > 1$, the mapping $p \mapsto p^{1/a}$ maps p into the unit ball of the Lebesgue space $L^a(\Omega, \mathcal{F}, \mu)$. Other parametrization are $\kappa = 1/a$ (Kaniadakis) and $\alpha = (a-2)/a$ (Amari). The basic idea is to to pull-back the structure of the unit ball of L^a to construct a Banach manifold on the set of densities $\mathcal{M}_>$. The actual construction is not straightforward in the infinite dimensional case, because the image set of the Amari embedding has empty interior in the Lebesgue space because of the non-negativity constrain $p^{1/a} \geq 0$. However, the approach works perfectly well in the cases of either a parametric model or a general model over a finite state space. We are going to discuss what happens if we use $\ln_\kappa$ instead of the α-embedding. A variant of the Amari embedding has been studied by [18], cf. also [12]. The same problem has been considered by many authors in Statistics.

There is also literature covering the non-commutative (quantum) case which is not treated here, e.g. [19], [20], [21]. The related issue of the algebro-geometric aspect of statistical models is briefly discussed, see the books [22–24] on Algebraic Statistics and [25] on the relations between Algebraic Statistics and Information Geometry.

The next Section is intended to be an introduction to both the algebraic and the geometric features of deformed exponential statistical models.

2 κ-Deformed Gibbs model

On a finite state space Ω, equipped with the energy function $U\colon \Omega \to \mathbb{R}_\geq$, we want to discuss the κ-deformation of the standard Gibbs model. There are two options, related with two different presentation of the normalizing constant (partition function).

The first option is to consider the statistical model

$$p(x;\theta) = \frac{\exp_\kappa(\theta U(x))}{Z(\theta)} = \exp_\kappa\left(\theta U(x) \overset{\kappa}{\oplus} \ln_\kappa\left(\frac{1}{Z(\theta)}\right)\right) \tag{27}$$

or, with $\widetilde{\psi}_\kappa(\theta) = \ln_\kappa Z(\theta)$,

$$\ln_\kappa p(x;\theta) = \theta U(x)\sqrt{1+\kappa^2(\widetilde{\psi}_\kappa(\theta))^2} - \widetilde{\psi}_\kappa(\theta)\sqrt{1+\kappa^2\theta^2U(x)^2} \tag{28}$$

We will not discuss this model here.

The second option is to define the model as

$$p(x;\theta) = \exp_\kappa(\theta U(x) - \psi_\kappa(\theta)) = \exp_\kappa(\theta U(x)) \overset{\kappa}{\otimes} \exp_\kappa(-\psi_\kappa(\theta)), \tag{29}$$

where $\psi_\kappa(\theta)$ is the unique solution of the equation

$$\sum_{x\in\Omega} \exp_\kappa(\theta U(x) - \psi_\kappa(\theta)) = 1. \tag{30}$$

Cf. the discussion in [9,10]. The derivative with respect to θ of the left hand side of (30) is

$$\sum_{x\in\Omega} \frac{U(x)-\psi_\kappa'(\theta)}{\sqrt{1+\kappa^2(\theta U(x) - \psi_\kappa(\theta))^2}} \exp_\kappa(\theta U(x) - \psi_\kappa(\theta)), \tag{31}$$

therefore

$$\mathrm{E}_\theta\left[\frac{U-\psi'(\theta)}{\sqrt{1+\kappa^2(\theta U - \psi_\kappa(\theta))^2}}\right] = 0, \tag{32}$$

where $\mathrm{E}_\theta[V] = \sum_x V(x)p(x;\theta)$. The one-parameter statistical models (27) and (29) are different unless $\kappa = 0$. This fact marks an important difference between the theory of ordinary exponential models and κ-deformed exponential models.

From the geometrical point of view, the second approach has the advantage of a the linear character of the model describing the $\ln_\kappa$-probability. Here, by geometry we mean differential geometry of statistical models, i.e. the construction of an atlas of charts representing the subset of probability densities unto open sets of a Banach space, see [26]. Before moving into this approach, we discuss the duality of the Gibbs model and its algebraic features.

Let $V = \mathrm{Span}(1, U)$ and $V^\perp$ the orthogonal space, i.e. $v \in V^\perp$ if, and only if, $\sum_x v(x) = 0$ and $\sum_x v(x)U(x) = 0$. It follows from Equation (29) that

$$\sum_{x\in\Omega} v(x)\ln_\kappa(p(x;\theta)) = 0, \qquad v \in V^\perp \tag{33}$$

Viceversa, if a strictly positive probability density function p is such that $\ln_\kappa p$ is orthogonal to $V^\perp$, then p belongs to the κ-Gibbs model for some θ.

For each $v \in V^\perp$, we can take its positive part v^+ and its negative part v^-, so that $v = v^+ - v^-$ and $v^+v^- = 0$. Equation (33) can be rewritten as

$$\sum_{x\colon v(x)>0} v^+(x)\ln_\kappa(p(x)) = \sum_{x\colon v(x)<0} v^-(x)\ln_\kappa(p(x)) \tag{34}$$

The interpretation of (34) is the following. As $\sum_x v(x) = 0$, we have

$$\sum_{x\in\Omega} v^+(x) = \sum_{x\in\Omega} v^-(x) = \lambda. \tag{35}$$

It follows that $r_1 = v^+/\lambda$, and $r_2 = v^-/\lambda$ are probability densities (states) with disjoint support, so that (34) can be restated by saying that a positive density p belongs to the κ-Gibbs model if, and only if,

$$\mathrm{E}_{r_1}[\ln_\kappa(p)] = \mathrm{E}_{r_2}[\ln_\kappa(p)] \tag{36}$$

for each couple of densities r_1, r_2 such that $r_1r_2 = 0$ and $\mathrm{E}_{r_1}[U] = \mathrm{E}_{r_2}[U]$, where $\mathrm{E}_r[V]$ denotes the mean value of V with respect to r, i.e. $\sum_x V(x)r(x)$.

If $v \in V^\perp$ happens to be integer valued, using the κ-algebra and the notation

$$\overbrace{x \overset{\kappa}{\otimes} \cdots \overset{\kappa}{\otimes} x}^{n \text{ times}} = x^{\overset{\kappa}{\otimes} n}, \tag{37}$$

we can write (34) as

$$\overset{\kappa}{\bigotimes}_{x\colon v(x)>0} p(x)^{\overset{\kappa}{\otimes} v^+(x)} = \overset{\kappa}{\bigotimes}_{x\colon v(x)<0} p(x)^{\overset{\kappa}{\otimes} v^-(x)}, \tag{38}$$

It should be noticed that (38) is continuous function of $p(x)$, $x \in \Omega$ and it does not require the strict positivity of each $p(x)$, $x \in \Omega$. Therefore, the same equation is satisfied by all limits (if any) of the model (29). Moreover, (38) is a *κ-polynomial invariant* for the κ-Gibbs model, cf. in [15] the discussion of the case $\kappa = 0$.

The set κ-polynomial equations of type (34) is not finite, because each equation depends on the choice of an integer valued vector v in the orthogonal space $V^\perp$. Accurate discussion of this issue requires tools from commutative algebra. If the energy function U takes its values on a lattice, we can choose integer valued random variables $v_1, \dots, v_{N-2}$ to be a linear basis of the orthogonal space $V^\perp$. In such a case, we have a finite system of binomial equations

$$\overset{\kappa}{\bigotimes}_{x\colon v_j(x)>0} p(x)^{\overset{\kappa}{\otimes} v_j^+(x)} = \overset{\kappa}{\bigotimes}_{x\colon v(x)<0} p(x)^{\overset{\kappa}{\otimes} v_j^-(x)}, \tag{39}$$

$j = 1, \dots, N-2$, which is equivalent to the original model (29).

In the classical case $\kappa = 0$, the polynomial invariants of the Gibbs model form a polynomial ideal I in the polynomial ring $\mathbb{Q}[p(x)\colon x \in \Omega]$. The ideal I admits, because of the Hilbert Theorem, a finite generating set. The discussion of various canonical form of such generating set is one of the issues of Algebraic Statistics.

Example We specialize our discussion to the toy example which is discussed in the case $\kappa = 0$ in [15]. Consider $\Omega = \{1,2,3,4,5\}$ and $U(1) = U(2) = 0$, $U(3) = 1$, $U(4) = U(5) = 2$. The following display shows a set of integer valued v_j, $j = 1,2,3$ of the orthogonal space $V^\perp$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & U & v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -4 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & -1 & 1 \end{bmatrix} \end{array} \tag{40}$$

Equation (39) becomes:

$$\begin{cases} p(1) = p(2) \\ p(4) = p(5) \\ p(1) \overset{\kappa}{\otimes} p(2) \overset{\kappa}{\otimes} p(4) \overset{\kappa}{\otimes} p(5) = p(3)^{\overset{\kappa}{\otimes} 4} \end{cases} \tag{41}$$

A strictly positive probability density belongs to the κ-Gibbs model (29) if and only if it satisfies (41). The set of all polynomial invariants of the Gibbs model is a polynomial ideal and the question of finding a set of generators is intricate. A non strictly positive density that is a solution of (41) is either $p(1) = p(2) = p(3) = 0$, $p(4) = p(5) = 1/2$, or $p(1) = p(2) = 1/2$, $p(3) = p(4) = p(5) = 0$. These two solutions are the uniform distributions on the sets of values that respectively maximize or minimize the energy function.

From (25) we can derive an other form of the last equation in the system (41):

$$\left(p^\kappa(1) - \frac{1}{p^\kappa(1)}\right) + \left(p^\kappa(2) - \frac{1}{p^\kappa(2)}\right) + \left(p^\kappa(4) - \frac{1}{p^\kappa(4)}\right) + \left(p^\kappa(5) - \frac{1}{p^\kappa(5)}\right) = 4\left(p^\kappa(3) - \frac{1}{p^\kappa(3)}\right).$$

A further algebraic presentation is available. In the model (29), we introduce the new parameters

$$\zeta_0 = \exp_\kappa\left(-\psi_\kappa(\theta)\right), \tag{42}$$
$$\zeta_1 = \exp_\kappa\left(\theta\right), \tag{43}$$

so that

$$p(x;\theta) = \exp_\kappa\left(\theta U(x)\right) \overset{\kappa}{\otimes} \exp_\kappa\left(-\psi_\kappa(\theta)\right), \tag{44}$$
$$= \zeta_0 \overset{\kappa}{\otimes} \zeta_1^{\overset{\kappa}{\otimes} U(x)}. \tag{45}$$

The probabilities are κ-monomials in the parameters ζ_0, ζ_1, e.g.:

$$\begin{cases} p(1) = p(2) = \zeta_0 \\ p(3) = \zeta_0 \overset{\kappa}{\otimes} \zeta_1 \\ p(4) = p(5) = \zeta_0 \overset{\kappa}{\otimes} \zeta_1^{\overset{\kappa}{\otimes} 2} \end{cases} \tag{46}$$

In algebraic terms, such a model is called a *toric model*. It is interesting to note that in (46) the parameter ζ_0 is required to be strictly positive, while the parameter ζ_1 could be zero, giving rise the uniform distribution on $\{1,2\} = \{x \colon U(x) = 0\}$. The other limit solution is not obtained by (46). The algebraic elimination of the indeterminates ζ_0, ζ_1 in (46) will produce back polynomial invariants. For example, from $(\zeta_o\zeta_1)^2 = (\zeta_0)(\zeta_0\zeta_1^2)$, we get $p(3)^2 = p(2)p(5)$. The (non-)uniqueness issue of the monomial parametric representation (46), together with the fact that one of the limit solution is not represented is discussed in [15]. If $\kappa \neq 0$, the notion of toric model and the related discussion could apply to the other model (27) which we are not discussing here.

3 Charts

We use the coordinate-free formalism of differential geometry of [26]. Let us fix a $\kappa \in]0,1[$. In order to construct an atlas of charts, i.e. mappings from the set of strictly positive probability densities $\mathcal{M}_>$ to some model vector space, we consider each chart as associated to a density $p \in \mathcal{M}_>$. Such a p is used as a reference for every other density q of a suitable subset of $\mathcal{M}_>$, via the statistical notion of likelihood q/p.

We first define a κ-divergence. If $q, p \in \mathcal{M}_>$ satisfy the condition

$$\left(\frac{q}{p}\right)^\kappa, \left(\frac{p}{q}\right)^\kappa \in L^1(p), \tag{47}$$

then the κ-divergence is defined to be

$$D_\kappa(p\|q) = \mathrm{E}_p\left[\ln_\kappa\left(\frac{p}{q}\right)\right] = \frac{1}{2\kappa}\mathrm{E}_p\left[\left(\frac{p}{q}\right)^\kappa - \left(\frac{q}{p}\right)^\kappa\right]. \tag{48}$$

The first of the two conditions in (47) is always satisfied because

$$\mathrm{E}_p\left[\left(\frac{q}{p}\right)^\kappa\right] \le \left(\mathrm{E}_p\left[\frac{q}{p}\right]\right)^\kappa = 1. \tag{49}$$

The second condition is non-trivial unless the state space is finite, because we are assuming

$$\mathrm{E}_p\left[\left(\frac{p}{q}\right)^\kappa\right] = \int \frac{p^{\kappa+1}}{q}\, d\mu < +\infty. \tag{50}$$

When such a condition is not satisfied the value of the expectation in (48) is $+\infty$. We are not interested in this case here.

The strict convexity of $-\ln_\kappa$ implies

$$D_\kappa(p\|q) = \mathrm{E}_p\left[-\ln_\kappa\left(\frac{q}{p}\right)\right] \ge -\ln_\kappa\left(\mathrm{E}_p\left[\frac{q}{p}\right]\right) = \ln_\kappa(1) = 0. \tag{51}$$

with equality if, and only if $q = p$.

The manifold we want to define has to be modelled on the Lebesgue space of centered $1/\kappa$-p-integrable random variables $L_0^{1/\kappa}(p)$, i.e. $v \in L_0^{1/\kappa}(p)$ if $\int |v|^{1/\kappa} p \, d\mu < +\infty$ and $\int vp \, d\mu = 0$. At each p there is a different model space, so that an isomorphism between them has to be provided. The simplest isometric identification between $L^{1/\kappa}(p_1)$ and $L^{1/\kappa}(p_2)$ is

$$L^{1/\kappa}(p_1) \ni u \mapsto \left(\frac{p_1}{p_2}\right)^\kappa u \in L^{1/\kappa}(p_2) \tag{52}$$

In fact,

$$\int \left| \left(\frac{p_1}{p_2}\right)^\kappa u \right|^{1/\kappa} p_2 \, d\mu = \int |u|^{1/\kappa} p_1 \, d\mu. \tag{53}$$

We use a variation of the formalism used in the exponential case [11]. The new manifold will be called *κ-statistical manifold*. We define the subset $\mathcal{E}_p$ of $\mathcal{M}_>$ by

$$\begin{aligned}\mathcal{E}_p &= \left\{ q \in \mathcal{M}_> : \left(\frac{q}{p}\right)^\kappa, \left(\frac{p}{q}\right)^\kappa \in L^{1/\kappa}(p) \right\} \\ &= \left\{ q \in \mathcal{M}_> : \frac{q}{p}, \frac{p}{q} \in L^1(p) \right\} \\ &= \left\{ q \in \mathcal{M}_> : \frac{p}{q} \in L^1(p) \right\}\end{aligned} \tag{54}$$

The assumption in (54) is stronger than that in (50), because here we want $\int p^2/q \; d\mu < +\infty$. Indeed, we are assuming more than the mere existence of $D_\kappa(p\|q)$, unless the state space is finite. It should be noticed that the set $\mathcal{E}_p$ does not depend on $\kappa \in]0,1[$. The condition $\int p^2/q \; d\mu < +\infty$ can be interpreted with the aid of the isometry (52). As

$$\left(\frac{p}{q}\right)^\kappa \in L^{\frac{1}{\kappa}}(p) \tag{55}$$

by definition, we can map it to $L^{\frac{1}{\kappa}}(q)$ by multiplying it by itself, to get

$$\left(\frac{p}{q}\right)^{2\kappa} \in L^{\frac{1}{\kappa}}(q), \quad \text{or} \quad \left(\frac{p}{q}\right)^2 \in L^1(q). \tag{56}$$

Each of the sets $\mathcal{E}_p$, $p \in \mathcal{M}_>$, is going to be the domain of a chart, and this will define an atlas of charts because of the covering $\mathcal{M}_> = \cup_p \mathcal{E}_p$; a connected component of the manifold will be the union of overlapping $\mathcal{E}$'s.

If $q \in \mathcal{E}_p$, then q is almost surely positive and we can write it in the form $q = \exp_\kappa(v) \cdot p$, where

$$v = \ln_\kappa\left(\frac{q}{p}\right) = \frac{\left(\frac{q}{p}\right)^\kappa - \left(\frac{p}{q}\right)^\kappa}{2\kappa} \in L^{1/\kappa}(p) \tag{57}$$

The expected value at p of $v = \ln_\kappa\left(\frac{q}{p}\right)$ is

$$\mathrm{E}_p\left[\ln_\kappa\left(\frac{q}{p}\right)\right] = -D_\kappa(p\|q), \tag{58}$$

so that we can write every $q \in \mathcal{E}_p$ as

$$q = \exp_\kappa\left(u - D_\kappa(p\|q)\right) \cdot p, \tag{59}$$

where u is a uniquely defined element of the set of centered $1/\kappa$-p-integrable random variables $L_0^{1/\kappa}(p)$, namely

$$u = \ln_\kappa\left(\frac{q}{p}\right) - \mathrm{E}_p\left[\ln_\kappa\left(\frac{q}{p}\right)\right] = \ln_\kappa\left(\frac{q}{p}\right) + D_\kappa(p\|q). \tag{60}$$

Conversely, given any $u \in L_0^{1/\kappa}(p)$, the real function $\psi \mapsto \mathrm{E}_p\left[\exp_\kappa(u - \psi)\right]$ is continuous and strictly decreasing from $+\infty$ to 0, therefore there exists a unique value of ψ, say $\psi_{\kappa,p}(u)$, such that

$$\mathrm{E}_p\left[\exp_\kappa\left(u - \psi_{\kappa,p}(u)\right)\right] = 1, \tag{61}$$

so that

$$q = \exp_\kappa\left(u - \psi_{\kappa,p}(u)\right) \cdot p \in \mathcal{E}_p \subset \mathcal{M}_>. \tag{62}$$

The 1-to-1 mapping

$$\mathcal{E}_p \ni q \leftrightarrow u \in L_0^{1/\kappa}(p) \tag{63}$$

is our chart. By comparing (62) with (59) we obtain $\psi_{\kappa,p}(u) = D_\kappa(p\|q)$, where u is the image of q in the chart at p. I.e. the p-chart representation of the functional $q \mapsto D_\kappa(p\|q)$ is $\psi_{\kappa,p}$.

The functional $\psi_{\kappa,p}$ is of key importance in the case $\kappa = 0$, where it is the cumulant functional of the random variable u:

$$\psi_p(u) = \ln \mathrm{E}_p\left[\mathrm{e}^u\right] \tag{64}$$

The proof of differentiability properties, in the non-finite case with $\kappa = 0$ is not trivial, see e.g. [14], because the domain of ψ_p functional has a non-trivial description in that case. For $\kappa \neq 0$, the domain of $\psi_{\kappa,t}$ is the full space $L_0^{1/\kappa}(p)$, and we can compute the directional derivatives

$$D\psi_{\kappa,p}(u)v = \left.\frac{d}{dt}\psi_{\kappa,p}(u + tv)\right|_{t=0} \tag{65}$$

and

$$D^2\psi_{\kappa,p}(u)vw = \left.\frac{d^2}{dsdt}\psi_{\kappa,p}(u + sv + tw)\right|_{s=t=0}. \tag{66}$$

For $\kappa = 0$, the Fréchet derivatives of ψ_p are

$$D\psi_p(u)v = \mathrm{E}_q\left[v\right], \tag{67}$$

$$D^2\psi_p(u)vw = \mathrm{Cov}_q\left(v, w\right), \tag{68}$$

where v and w are the directions of derivation and $q = \mathrm{e}^{u - \psi_p(u)} \cdot p$.

For $\kappa \neq 0$ the computation of the directional derivative (65) of (61) gives

$$\mathrm{E}_p\left[\exp_\kappa{}'\left(u - \psi_{\kappa,p}(u)\right)\left(v - D\psi_{\kappa,p}(u)v\right)\right] = \mathrm{E}_q\left[\frac{v - D\psi_{\kappa,p}(u)v}{\sqrt{1 + \kappa^2(u - \psi_{\kappa,p}(u))^2}}\right] = 0. \tag{69}$$

It follows from (69) with $u=0$ that $D\psi_{\kappa,p}(0)=0$.

Otherwise, let $q|p$ denote the density proportional to

$$\exp_\kappa{}'(u-\psi_{\kappa,p}(u))\cdot p = \frac{q}{\sqrt{1+\kappa^2(u-\psi_{\kappa,p}(u))^2}} = \frac{q}{\sqrt{1+\kappa^2\left(\ln_\kappa\left(\frac{q}{p}\right)\right)^2}}, \quad (70)$$

see [10], where such a density is called escort probability. The explicit expression for the derivative is

$$D\psi_{\kappa,p}(u)v = \mathrm{E}_{q|p}\left[v\right], \quad (71)$$

which is the same as (67), but the expectation is computed with respect of the escort density $q|p$. Later we will give a geometric interpretation of $q|p$.

The second derivative of $u \mapsto \exp_\kappa(u-\psi_{\kappa,p}(u))$ in the directions v and w is the first derivative in the direction w of $u\mapsto \exp_\kappa{}'(u-D\psi_{\kappa,p}(u))(v-D\psi_{\kappa,p}(u)v)$, therefore it is equal to

$$\exp_\kappa{}''(u-\psi_{\kappa,p}(u))(v-D\psi_{\kappa,p}(u)v)(w-D\psi_{\kappa,p}(u)w) - \exp_\kappa{}'(u-D\psi_{\kappa,p}(u))D^2\psi_{\kappa,p}(u)vw. \quad (72)$$

The random variable in (72) has zero p-expectation, so that

$$D^2\psi_{\kappa,p}(u)vw = \frac{\mathrm{E}_p\left[\exp_\kappa{}''(u-\psi_{\kappa,p}(u))(v-D\psi_{\kappa,p}(u)v)(w-D\psi_{\kappa,p}(u)w)\right]}{\mathrm{E}_p\left[\exp_\kappa{}'(u-D\psi_{\kappa,p}(u))\right]}. \quad (73)$$

If $w=v\neq 0$, then $D^2\psi_{\kappa,p}(u)vv>0$, therefore the functional $\psi_{\kappa,p}$ is strictly convex. For $u=0$ we obtain

$$D^2\psi_{\kappa,p}(0)vw = \mathrm{Cov}_p\left(u,v\right). \quad (74)$$

We do not have a similar interpretation for $u\neq 0$, but see the discussion of parallel transport below.

4 The κ-statistical manifold and its tangent bundle

Assume now we want to change of chart, that is we want to change the reference density from p to $\bar p$ to represent a q that belongs both to $\mathcal{E}_p$ and to $\mathcal{E}_{\bar p}$. From now on, we skip the discussion of the non-finite case; such a discussion will be published elsewhere. In the finite state space case, the integrability conditions of (54) are always satisfied, so that all the chart's domains $\mathcal{E}_p$ are equal to $\mathcal{M}_>$. The application of (60) and (62) to the change

$$L_0^{1/\kappa}(p) \ni u \mapsto q \mapsto \bar u \in L_0^{1/\kappa}(\bar p) \quad (75)$$

gives

$$\begin{aligned}\bar u &= \ln_\kappa\left(\frac{q}{\bar p}\right) - \mathrm{E}_{\bar p}\left[\ln_\kappa\left(\frac{q}{\bar p}\right)\right] \\ &= \ln_\kappa\left(\exp_\kappa\left(u-\psi_{\kappa,p}(u)\right)\frac{p}{\bar p}\right) \\ &\quad - \mathrm{E}_{\bar p}\left[\ln_\kappa\left(\exp_\kappa\left(u-\psi_{\kappa,p}(u)\right)\frac{p}{\bar p}\right)\right] \\ &= (u-\psi_{\kappa,p}(u)) \overset{\kappa}{\oplus} \ln_\kappa\left(\frac{p}{\bar p}\right) \\ &\quad - \mathrm{E}_{\bar p}\left[(u-\psi_{\kappa,p}(u)) \overset{\kappa}{\oplus} \ln_\kappa\left(\frac{p}{\bar p}\right)\right]\end{aligned} \quad (76)$$

For $\kappa=0$ the change of chart is an affine function:

$$\bar u = u + \ln\left(\frac{p}{\bar p}\right) - \mathrm{E}_{\bar p}\left[u+\ln\left(\frac{p}{\bar p}\right)\right] \quad (77)$$

with linear part

$$v \mapsto v - \mathrm{E}_{\bar p}\left[v\right]. \quad (78)$$

For $\kappa\neq 0$, the derivative in the direction v of the change of chart at u is obtained from the derivation formula (21). It has the form $A-\mathrm{E}_{\bar p}\left[A\right]$, with

$$A = \left(\sqrt{1+\kappa^2\left(\ln_\kappa\left(\frac{p}{\bar p}\right)\right)^2} + \frac{\kappa^2\ln_\kappa\left(\frac{p}{\bar p}\right)(u-\psi_{\kappa,p}(u))}{\sqrt{1+\kappa^2\left(u-\psi_{\kappa,p}(u)\right)^2}}\right) \times (v-D\psi_{\kappa,p}(u)v). \quad (79)$$

We are then led to the study the tangent spaces of the κ-statistical manifold. Let p_θ, $\theta\in]-1,1[$, be a curve in $\mathcal{E}_p$,

$$p_\theta = \exp_\kappa\left(u_\theta - \psi_{\kappa,p_0}(u_\theta)\right)\cdot p_0. \quad (80)$$

In the chart at p the velocity vector is given by

$$\dot{u_\theta} \in L_0^{1/\kappa}(p). \quad (81)$$

We identify the tangent space at p_0 with the space of the random variables $\dot{u_0}$. In general, the tangent space at $p\in\mathcal{M}_>$ is defied to be $T_p = L_0^{1/\kappa}(p)$. Derivation with respect to θ of (80) gives

$$\frac{\dot{p_\theta}}{p_\theta} = (1+\kappa^2(u_\theta-\psi_{\kappa,p}(u_\theta)^2)^{-1/2}(\dot{u_\theta} - D\psi_{\kappa,p}(u_\theta)(\dot{u_\theta})). \quad (82)$$

In particular, $\dot{p_0}/p_0 = \dot{u_0}$. To extend this to the general θ, let us observe first that (69), with $p=p_0$, $u=u_\theta$, $v=\dot{u_\theta}$, shows that $\mathrm{E}_{p_0}\left[\dot{p_\theta}/p_\theta\right]=0$. The second factor in the right end side of (82) is, because of (71),

$$\dot{u_\theta} - D\psi_{\kappa,p}(u_\theta)(\dot{u_\theta}) = \dot{u_\theta} - \mathrm{E}_{p_\theta|p_0}\left[\dot{u_\theta}\right]. \quad (83)$$

The first factor is

$$\left(1+\kappa^2(u_\theta-\psi_{\kappa,p}(u_\theta)^2\right)^{-1/2} = \left(1+\kappa^2\left(\ln_\kappa\left(\frac{\dot{p_\theta}}{p_0}\right)\right)^2\right)^{-1/2} \quad (84)$$

Let us define a parallel transport $U^\kappa_{p,\bar p}$ mapping the tangent space at p, i.e. $T_p = L_0^{1/\kappa}(p)$, on the tangent space at $\bar p$, i.e. $T_{\bar p} = L_0^{1/\kappa}(\bar p)$ as follows. If $u \in L_0^{1/\kappa}(p)$, then the random variable

$$\bar u = \frac{u - \mathrm{E}_{\bar p|p}[u]}{\sqrt{1 + \kappa^2 \left(\ln_\kappa\left(\frac{\bar p}{p}\right)\right)^2}} \tag{85}$$

belongs to $T_{\bar p} = L_0^{1/\kappa}(\bar p)$, and we define $\bar u = U^\kappa_{p,\bar p}(\bar u)$. In fact, $\mathrm{E}_{\bar p}[\bar u] = 0$, and, moreover,

$$\left(1 + \kappa^2 \left(\ln_\kappa\left(\frac{\bar p}{p}\right)\right)^2\right)^{-1/2} \leq 2\left(\frac{\bar p}{p}\right)^\kappa, \tag{86}$$

so that the conclusion

$$\mathrm{E}_{\bar p}\left[|U^\kappa_{p,\bar p}(u)|^{1/\kappa}\right] \leq 2^{1/\kappa} \mathrm{E}_p\left[|u|^{1/\kappa}\right] \tag{87}$$

follows from the isometry (52).

The quantity $\dot{p_\theta}/p_\theta$ in (82) has the following remarkable interpretation in terms of the parallel transport:

$$\frac{\dot{p_\theta}}{p_\theta} = U^\kappa_{p_0,p_\theta}(\dot{u_\theta}), \tag{88}$$

i.e. it is the velocity vector at θ, transported to the tangent space at p_θ.

In this setting, we can define κ-exponential models in a non parametric way as

$$q = \exp_\kappa\left(u - \psi_{\kappa,p}(u)\right) \cdot p, \quad u \in V, \tag{89}$$

where V is a linear sub-space of $L_0^{1/\kappa}(p)$. Each $v \in V$ is called a canonical variable of the κ-exponential model. The implicit representation of the exponential model (89) is

$$\mathrm{E}_p\left[\ln_\kappa\left(\frac{q}{p}\right) v\right] = 0, \quad v \in V^\perp. \tag{90}$$

where $V^\perp \subset L_0^{1-1/\kappa}(p)$ is the orthogonal of V. In Statistics, a $v \in V^\perp$ is called a constrain of the log-linear model. We could derive, as we did in the κ-Gibbs model example, for lattice-valued constrain variables v, the relevant polynomial-type equations based of the deformed product operation $\overset{\kappa}{\otimes}$.

In particular, a one dimensional exponential model is characterised by a one-dimensional space $V = \mathrm{Span}(u)$, $u \in L_0^{1/\kappa}(p)$. Let us show that such a model satisfies a differential equation on the manifold. Given $u \in L_0^{1/\kappa}(p_0)$, for each $q \in \mathcal{E}_p$, we can define the mapping

$$\mathcal{E}_p \ni q \mapsto U^\kappa_{p,q}(u) \in T_q, \tag{91}$$

which is a vector field of the κ-manifold. The velocity of a curve $p_\theta = \exp_\kappa\left(u_\theta - \psi_{\kappa,p_0}(u_\theta)\right) \cdot p_0$ is represented in the chart at p_θ by $\dot{p_\theta}/p_\theta$ because of (88), therefore we can consider the differential equation

$$\frac{\dot{p_\theta}}{p_\theta} = U^\kappa_{p_0,p_\theta}(u), \tag{92}$$

whose solution is

$$p_\theta = \exp_\kappa\left(\theta u - \psi_{\kappa,p}(\theta u)\right) \cdot p_0, \tag{93}$$

cf. (82).

The last one was just the simplest example, but the treatment of evolution equations for densities is one of the main motivation for introducing a manifold structure on the set of densities, see e.g. [27], [28], [29].

5 Statistical manifolds

In this final section, we go back to the general setting and briefly discuss how the κ-statistical manifolds we have defined relate with the previous construction of the exponential statistical manifold. $(\Omega, \mathcal{F}, \mu)$ is a generic probability space, $\mathcal{M}^1$ is the set of real random variables f such that $\int f\, d\mu = 1$, $\mathcal{M}_\geq$ the convex set of probability densities, $\mathcal{M}_>$ the convex set of strictly positive probability densities:

$$\mathcal{M}_> \subset \mathcal{M}_\geq \subset \mathcal{M}^1 \tag{94}$$

In the classical case, i.e. $\kappa = 0$, differentiable manifolds are defined on both $\mathcal{M}_>$ and $\mathcal{M}^1$. Such manifolds are both modeled on suitable Orlicz spaces, see [30]. Orlicz spaces are a generalization of Lebesgue spaces, where the norm is defined through a symmetric, null at zero, non-negative, convex, with more than linear growth, function called Young function. Let Φ be any Young function with growth equivalent to exp, e.g. $\Phi(x) = \cosh(x) - 1$, with convex conjugate Ψ, e.g. $\Psi(y) = (1+|y|)\log(1+|y|) - |y|$. The relevant Orlicz spaces are denoted by L^Φ and L^Ψ, respectively. Both these Banach spaces appear naturally in Statistics. A random variable u belongs to the space L^Φ if, an only if, its Laplace transform exists in a neighborhood of 0, that is, the one dimensional exponential model $p(\theta) \propto \mathrm{e}^{\theta u}$ is defined for values of the parameter in a neighborhood of 0. A density function f has finite entropy if, and only if, it belongs to the space L^Ψ. We denote by L_0^Φ, L_0^Ψ the sub-spaces of centered random variables. If the sample space is not finite, then the exponential Orlicz space is not separable and the closure M^Φ of the space of bounded functions is different from L^Φ. There is a natural separating duality between L_0^Φ and L_0^Ψ, which is given by the bi-linear form

$$(u, v) \mapsto \int uv\, d\mu. \tag{95}$$

It can be proved that the cumulant generating functional $\psi_p(u) = \mathrm{E}_p[\mathrm{e}^u]$, $u \in \mathrm{L}_0^\Psi(p)$, $p \in \mathcal{M}_>$ is positive, strictly convex, analytic. The interior of the proper domain of ψ_p,

$$\mathcal{S}_p = \left\{u \in \mathrm{L}_0^\Psi(p) : \psi_p(u) < +\infty\right\}^\circ \tag{96}$$

defines the so called maximal exponential model

$$\mathcal{E}_{0,p} = \left\{\mathrm{e}^{u - \psi_p(u)} \cdot p : u \in \mathcal{S}_p\right\} \tag{97}$$

In our case, i.e. $0 < \kappa < 1$, we could consider

$$\cosh_\kappa(x) - 1 = \frac{1}{2}(\exp_\kappa(x) + \exp_\kappa(-x))$$
$$= \frac{1}{2}\left(\mathrm{e}^{\int_0^x \frac{dt}{\sqrt{1+\kappa^2t^2}}} + \mathrm{e}^{\int_0^{-x} \frac{dt}{\sqrt{1+\kappa^2t^2}}}\right) - 1$$
$$= \cosh\left(\int_0^x \frac{dt}{\sqrt{1+\kappa^2t^2}}\right) - 1. \quad (98)$$

We obtain a Young function equivalent to $|x|^{1/k}$, so that the related Orlicz space is just $L^{1/\kappa}$ and the κ-statistical manifolds appears as a generalisation of the exponential construction.

If $q \in \mathcal{E}_{0,p}$, then $q = \exp(u - \psi_p(u)) .p$. If $-u \in \mathcal{S}_p$, then $q \in \mathcal{E}_p$, so that $q = \exp_\kappa(v - \psi_{k,p}(v))$ for a suitable $v \in L_0^{1/\kappa}(p)$. It follows that

$$\exp(u - \psi_p(u)) = \exp_\kappa(v - \psi_{k,p}(v)), \quad (99)$$

therefore the e-coordinate u and κ-coordinate v are related by the equation

$$u - \psi_p(u) = \int_0^{v-\psi_{k,p}(v)} \frac{dt}{\sqrt{1+\kappa^2t^2}}. \quad (100)$$

6 Conclusion

We have presented a non parametric construction of the statistical manifold based on the use of the centered $\ln_\kappa$-likelihood as a functional coordinate. The first steps of the derivation are discussed in Section 3 and 4. A notable difference from the standard case $\kappa = 0$ is the absence of simple formulæ for the functional $\psi_{\kappa,p}$ and its derivatives. It has been shown that the derivation of such quantities is related with parallel transport of the tangent bundle, which, in turn, should lead to a theory of evolution equations on the κ-manifold. The algebraic features of the κ-exponential models for lattice contrasts have been described on an example. In comparison with similar construction of the statistical manifold based on other transformations derived from the Box-Cox transform, it should be noticed that the κ-logarithm has a distinctive advantage of having range $\mathbb{R}$ while retaining a simple algebraic character.

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