PROBABILITY 2020
EXAM

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1. Bernoulli probability function

Let X be a finite set, $\#X = n$. We define a probability function on the finite set of parts $S = \mathcal{P}(X)$, $\#S = 2^n$. If $s \in S$, that is, $s \subset X$, denote by the same letter both the set and its indicator function, $s(x) = 1$ if $x \in s$, 0 otherwise. One can check that for each $p \in]0,1[$,

$$p(s) = \prod_{x \in X} p^{s(x)}(1-p)^{1-s(x)} = p^{\#s}(1-p)^{n-\#s} = (1-p)^n \left(\frac{p}{1-p}\right)^{\#s}$$

is a probability function. It is the *Bernoulli* probability function.

For the corresponding probability measure it holds $\mathbb{P}_p\left(\#s = k\right) = \binom{n}{k} p^k (1-p)^{n-k}$. The equation $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ defines the *binomial* probability function on $\{0,1,\dots,n\}$.

Let $X = X_1 \cup X_2$ be a partition of X. We have

$$\mathbb{P}_p\left(s \colon s \subset X_1\right) = \sum_{s \subset X_1} p^{\#s}(1-p)^{n-\#s} =$$
$$\sum_{s \subset X_1} p^{\#s}(1-p)^{\#X_1-\#s}(1-p)^{\#X-\#X_1} = (1-p)^{\#X_2} \ .$$

For each $s \subset X$ define $s_1 = s \cap X_1$ and $s_2 = s \cap X_2$. Write $n_1 = \#X_1$, $n_2 = \#X_2$. The Bernoully probability function is decomposed as

$$p(s) = (1-p)^{n_1} \left(\frac{p}{1-p}\right)^{\#s_1} \times (1-p)^{n_2} \left(\frac{p}{1-p}\right)^{\#s_2} = p_1(s_1)p_2(s_2) \ .$$

[To be continued]

2. Random Graph

Let V be a finite set of *vertices*, $\#V = v$. The set of *edges* is the set of all 2-subsets of V. The usual notation is $\mathcal{E}$, but here we use $X = \binom{V}{2}$, $\#X = \binom{v}{2}$, for coherence with the previous section. A *graph* g is a set of edges. Let us consider the sample space of all graphs $G = \mathcal{P}(X)$, $\#G = 2^{\binom{v}{2}}$ and the Bernoully probability,

$$p(g) = (1-p)^{\binom{v}{2}} \left(\frac{p}{1-p}\right)^{\#g} \ .$$

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All graphs with the same number of edges have the same probability. Under this model, the probability that a graph has e edges, $0 \leq e \leq \binom{v}{2}$ is

$$\binom{\binom{v}{2}}{e}(1-p)^{\binom{v}{2}}\left(\frac{p}{1-p}\right)^{e} \ .$$

Let us decompose the vertex set into $V = V_1 \cup V_2$, $v = v_1 + v_2$. This procuces a partition of the set of edges into 3 parts, $X = X_1 \cup X_2 \cup X_{12}$. X_1 are the edges in V_1, X_2 are the edges in V_2, X_{12} are the edges connecting V_1 and V_2. Note that

$$\#X = \binom{v_1+v_2}{2}, \#X_1 = \binom{v_1}{2}, \#X = \binom{v_2}{2}, \#X_{12} = v_1 v_2 \ .$$

Every graph is decomposed as $g = g_1 \cup g_2 \cup g_{12}$ and

$$p(g) = p_1(g_1) \times p_2(g_2) \times (1-p)^{v_1 v_2}\left(\frac{p}{1-p}\right)^{\#g_{12}} \ .$$

[To be continued]

3. Monotone classes

Aet S be the sample space. A family $\mathcal{M}$ of subsets of S is a *monotone class* if for all infinite sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, if the sequence is non-increasing $A_n \supset A_{n+1}$ then $\lim_{n\to\infty} A_n = \cap_n A_n \in \mathcal{M}$, and, if the sequence is non-decreasing $A_n \subset A_{n+1}$ then $\lim_{n\to\infty} A_n = \cup_n A_n \in \mathcal{M}$.

If $\mathcal{B}$ is a field on S, then the σ-algebra generated by $\mathcal{B}$, $\sigma(\mathcal{B})$, coincides with the monotone class $m(\mathcal{B})$ generated by $\mathcal{B}$

- Consider the relation $\Phi(A, B)$ defined on the parts of of S by

  $$A \cup B, A \setminus B, B \setminus A \in m(\mathcal{B}) \ .$$

  For all A, $\Phi(A)$ is a monotone class.

[To be continued]