Stochastic Calculus 2013
Part 1

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Texbook

- The course is based on (part of) Chapters 4 to 6 of the textbook by Steven S Shreve *Stochastic Calculus for Finance II Continuous-Time Models* 2nd ed 2004 Springer.
- Prerequisite are Martingale and Brownian motion covered in the previous lectures and in Chapters 1 to 3 of the Shreve's textbook.
- In Shrive's textbook the mathematical treatment is essentialy rigorous, but most of the proofs are actually skipped in favor of the computations and their financial meaning. Such missing details are to be found in other textbooks, see the list in §4.9.

Brownian motion

Definition

W is a *Brownian motion* for $(\Omega, \mathcal{F}, \mathrm{P}, (\mathcal{F}(t)_{t\geq 0})$ if

- W is a continuous process, $W \colon \Omega \to C(\mathbb{R}_{>0})$, such that
- W is adapted, i.e. W_t is $\mathcal{F}_t$-measurable, $t > 0$,
- W starts from 0, i.e. $W_0 = 0$ a.s.,
- the increments are gaussian, precisely $W_t - W_s \sim \mathrm{N}(0, t-s)$, $0 \leq s < t$,
- the increments are independent from the past history, i.e. $W_t - W_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t$.

Piecewise linear approximation

Choose $\Delta > 0$ and the consider the discrete process $Z_n = \Delta^{1/2}(W_{n\Delta} - W_{(n-1)\Delta})$, $n = 1, 2, \dots$. Then Z_n is a gaussian white noise. The Brownian motion is approzimated by a piecewise linear interpolation of a suiably scaled random walk.

Properties of W

Theorem

Let $t_0 = 0 < t_1 < \cdots < t_n$ be a partition of the times.

- *The random variables $W_{t_1}, (W_{t_2} - W_{t_1}), \dots (W_{t_n} - W_{t_{n-1}})$ are independent.*
- *The vector $(W_{t_1}, \dots, W_{t_n})$ has density*

$$p(y_1, \dots, y_n) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} (y_j - y_{j-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(y_{t_j} - y_{t_{j-1}})^2}{t_j - t_{j-1}}\right)$$

- *W is a Markov process with kernel*

$$k(x, y) = \frac{1}{2\pi\sqrt{t-s}} \exp\left(-\frac{1}{2} \frac{(y-x)^2}{t-s}\right)$$

- *W is a martingale.*
- *$(W_t^2 - t)_{t>0}$ is a martingale.*

Continuous L^2 martingales

Definition

A continuous process M is an L^2 martingale if

1. $\mathbb{E}\left(M_t^2\right) < +\infty$, and
2. $\mathbb{E}\left(M_t \,|\mathcal{F}_s\right) = M_s, \quad 0 \leq s < t.$

Theorem

Let M_n, $n = 1, 2, \dots$ be a sequence of L^2 continuous martingales.Let T be a finite horizon and assume that the L^2 limit of $M_n(T)$ exists, i.e. there exists a random variable M such that

$$\lim_{n\to\infty} \mathbb{E}\left((M_n(T) - M)^2\right) = 0.$$

Then there exist an L^2 continuous martingale M_t, $t \in [0, T]$ such that

$$\lim_{n\to\infty} \mathbb{E}\left(sup_{t\in[o,T]}(M_n(t) - M_t)^2\right) = 0.$$

Simple processes

- W is a *Brownian motion* for $(\Omega, \mathcal{F}, \mathrm{P}, (\mathcal{F}(t)_{t\geq 0})$.
- An adapted process Δ is of class L^2 if

$$\mathbb{E}\left(\int_0^T \Delta^2(t)dt\right) < +\infty \quad \text{for all } T > 0.$$

- The set of adapted processes Δ of class L^2 is a vector space.
- If Y_1 is $\mathcal{F}(t_1)$ measurable and $\mathbb{E}\left(Y_1^2\right) < +\infty$, then the process $\Delta(t) = Y_1(t_1 \leq t)$ is adapted and of class L^2. A finite sum of such processes is a *simple process*:

$$\Delta(t) = \sum_{j=1}^n Y_j(t_j \leq t).$$

- All trajectory of a simple process are pure jumps and right-continuous. If we order the t_j's in increasing order, $t_1 < t_2 < \dots < t_n$, and $t_j \leq t < t_{j+1}$, then $\Delta_t = \sum_{i=1}^j Y_i$.

Ito integral of the simple process $\Delta(t) = Y_1(t_1 \leq t)$, $t \geq 0$

- For $\Delta(t) = Y_1(t_1 \leq t)$, define the *Ito integral*

$$\int_0^t \Delta(s)dW(s) = \begin{cases} 0 & \text{for } t < t_1 \\ Y_1(W(t) - W(t_1)) & \text{for } t_1 \leq t \end{cases}$$
$$= Y_1(W(t) - W(t \wedge t_1))$$

- The Ito integral is a *continuous martingale*

$$\mathbb{E}\left(\int_0^t \Delta(u)dW(u)\middle|\mathcal{F}(s)\right) = \int_0^s \Delta(u)dW(u), \quad s \leq t.$$

- The Ito integral is *isometric*

$$\mathbb{E}\left(\left(\int_0^t \Delta(u)dW(u)\right)^2\right) = \mathbb{E}\left(\int_0^t \Delta^2(u)du\right).$$

- The *quadratic variation* of the Ito integral is $\int_0^t \Delta^2(s)ds$.

Ito integral of a simple process Δ

- For $\Delta(t) = \sum_{j=1}^n Y_j(t_j \leq t)$, define the *Ito integral* by linearity. If the interval $[0, t]$ contains the jumps $0 \leq t_1 < \cdots t_m \leq t$,

$$\begin{aligned}\int_0^t \Delta(s)dW(s) &= \sum_{j=1}^n Y_j(W(t) - W(t \wedge t_j)) \\ &= \sum_{j=1}^m Y_j(W(t) - W(t_j)) \\ &= \sum_{j=1}^m Y_j(\sum_{i=j+1}^m W(t_{i+1}) - W(t_i)) \\ &= \sum_{i=1}^m \Delta(t_i)(W(t_{i+1}) - W(t_i))\end{aligned}$$

- The Ito integral is a *continuous martingale*.
- The Ito integral is *isometric*.
- The *quadratic variation* of the Ito integral is $\int_0^t \Delta^2(s)ds$.

Ito integral of an L^2 process

- If Δ is a process of class L^2, there exists a sequence Δ_n, $n = 1, 2, \ldots$ of simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T |\Delta(u) - \Delta_n(u)|^2 \, du\right) = 0.$$

- The Ito integral of a process of class L^2 is defined by continuity.
- The Ito integral is a linear operator mapping L^2 processes into continuous martingale.
- The Ito integral is isometric.
- The quadratic variation of the Ito integral is

$$\left[\int \Delta dW\right](t) = \int_0^t \Delta^2(u) du$$

Continuous martingales

If M is a continuous bounded martingale, the computation

$$\begin{aligned} M^2(t) - M^2(s) &= \sum_{j=1}^n M^2(t_j) - M^2(t_{j-1}) \\ &= \sum_{j=1}^n 2M(t_{j-1})(M(t_j) - M(t_{j-1})) + \sum_{j=1}^n (M(t_j) - M(t_{j-1}))^2 \end{aligned}$$

produces the decomposition

$$M^2(t) = M^2(0) + 2\int_0^t M(u)dM(u) + [M](t)$$

and, for an Ito integral,

$$\left(\int_0^t \Delta dW\right) = 2\int_0^t \left(\int_0^s \Delta(u)dW(u)\right) dW(s) + \int_0^t \Delta^2(s)ds$$

Ito-Doeblin formula

Definition (Ito process)

An *Ito process* is a process of the form

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds.$$

Theorem (Ito-Doeblin formula for the Brownian Motion)

If

- $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^2)$ *and*
- $f_x(t, W(t))$, $t \geq 0$, *is an* L^2 *process,*

then $f(t, W(t))$, $t \geq 0$, *is an Ito process, and*

$$f(t, W(t)) = f(0, W(0)) + \int_0^t f_t(s, W(s))ds + \int_0^t f_x(s, W(s))dW(s) + \frac{1}{2}\int_0^t f_{xx}(s, W(s))ds.$$

Proof of Ito-Doeblin formula I

We write for $0 \leq s < t \leq T$

$$\begin{aligned} \int_s^t \Delta(u)dW(u) &= \int_0^t \Delta(u)dW(u) - \int_0^s \Delta(u)dW(u) \\ &= \int_0^T (s < u \leq t)\Delta(u)dW(u). \end{aligned}$$

In particular,

$$\begin{aligned} (W(t) - W(s))^2 &= W(t)^2 - W(s)^2 - 2W(s)(W(t) - W(s)) \\ &= 2\int_s^t W(u)dW(u) + (t - s) - 2W(s)(W(t) - W(s)) \\ &= (t - s) + 2\int_s^t (W(u) - W(s))dW(u) \end{aligned}$$

Proof of Ito-Doeblin formula II

The Taylor formula of order 1,2 for f gives

$$\begin{aligned} f(t,W(t)) - f(s,W(s)) =& f_t(s,W(s))(t-s) \\ &+ f_x(s,W(s))(W(t)-W(s)) \\ &+ \frac{1}{2} f_{xx}(s,W(s))(W(t)-W(s))^2 \\ &+ R_{1,2}(s,t,W(s),W(t)) \\ =& f_t(s,W(s))(t-s) \\ &+ f_x(s,W(s))(W(t)-W(s)) \\ &+ \frac{1}{2} f_{xx}(s,W(s))(t-s) \\ &+ f_{xx}(s,W(s)) \int_s^t (W(u)-W(s))dW(u) \\ &+ R_{1,2}(s,t,W(s),W(t)) \end{aligned}$$

Summing over a partition, the first tree term go to the Ito formula, the last two terms go to zero.

Ito-Doeblin formula: Applications I

- The process $f(t,W(t))$ is a martingale if $f_{10}(t,x) + \frac{1}{2} f_{02}(t,x) = 0$.
- Let $H_n(x)$ be a polynomial of degree n and define $f(t,x) = t^{n/2} H_n(t^{-1/2}x)$. We have

$$\begin{aligned} f_{1,0}(t,x) &= t^{n/2-1}\left(\frac{1}{2} H_n(t^{-1/2}x) - \frac{x}{2} H_n'(t^{-1/2}x)\right), \\ f_{02}(t,x) &= t^{n/2-1} H_n''(t^{-1/2}x). \end{aligned}$$

- The martingale condition is satified if

$$nH_n(y) - yH_n'(y) + H_n''(y) = 0.$$

Ito-Doeblin formula: Applications II

- We can take the *Hermite polynomials*

  $$H_n(y) = (-1)^n \mathrm{e}^{\frac{y^2}{2}} \frac{d^n}{dy^n} \mathrm{e}^{-\frac{y^2}{2}}$$

  to obtain the *Hermite martingales*

  $$M_n(t) = \int_0^t u^{\frac{n}{2}} H_n(u^{-\frac{1}{2}} W(u)) dW(u).$$

  [Hint: the n-th derivative of $yg(y)$ is $yg^{(n)}(y) + ng^{(n-1)}(y)$]
- As $H_n'(y) = nH_{n-1}(y)$, if $f_n(t,x) = t^{n/2} H_n(t^{-1/2}x)$, the x-derivative is

  $$\frac{d}{dx} f_n(t,x) = t^{\frac{n}{2}-\frac{1}{2}} H_n'(t^{-1/2}x) = nf_{n-1}(t,x),$$

  and we have the *iterated* Ito integrals

  $$M_n(t) = \int_0^t M_{n-1}(u) dW(u).$$

Ito processes

- For an Ito process $X(t) = X_0 + M(t) + A(t)$, $t \geq 0$, the integral is defined by approximation on simple processes.
- The M part and the A part behave differenthy when the quadratic variation is considered.
-

$$X^2(t) = X^2(0) + 2\int_0^t X(s)dX(s) + [M](t) =$$

$$X_0^2 + 2\int_0^t X(s)\Delta(s)dW(s) + 2\int_0^t X(s)\Theta(s)ds + \int_0^t \Delta^2(s)ds$$

- *The quadratic variation of X and the quadratic variation of M are equal.*

Ito-Doebin for Ito process

Theorem (Ito-Doeblin formula for the Ito process)

If

- $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$,
- X *is a Ito process with* $dX(t) = \Delta(t)dW(t) + \Theta(t)dt$,
- $f_x(t, X(t))\Delta(t)$, $t \geq 0$, *is an* L^2 *process,*

then $f(t, X(t))$, $t \geq 0$, *is an Ito process, and*

$$f(t, X(t)) =$$

$$f(0, X(0)) + \int_0^t f_t(s, X(s))ds + \int_0^t f_x(s, X(s))dX(s) + \frac{1}{2}\int_0^t f_{xx}(s, X(s))d[X](s)$$

$$= f(0, X(0)) + \int_0^t f_t(s, X(s))ds + \int_0^t f_x(s, X(s))\Delta(s)dW(s)$$

$$+ \int_0^t f_x(s, X(s))\Theta(s)ds + \frac{1}{2}\int_0^t f_{xx}(s, X(s))\Delta^2(s)ds$$

Geometric Brownian Motion

The process $f(t, W(t))$ is a martingale if $f_{10}(t, x) + \frac{1}{2}f_{02}(t, x) = 0$, for example

$$f(t, x) = \exp\left(\theta x - \frac{1}{2}\theta^2 t\right).$$

In such a case $f(0, 0) = 1$ and

$$f_{01}(t, x) = \theta f(t, x).$$

Definition (Geometric Brownian motion)

The process $X(t) = \exp\left(\theta W(t) - \frac{1}{2}\theta^2 t\right)$ is a *positive martingale* and

$$X(t) = 1 + \theta \int_0^t X(u)dW(u)$$

More generally, the process $X(t) = \exp\left(\int_0^t \theta(u)dW(u) - \frac{1}{2} \in_0^t \theta^2(u)du\right)$ is a positive martingale and $dX(t) = \theta(t)X(t)dW(t)$.

Vasicek interest rate model, Example 4.4.10

The solution of the *stochastic differential equation SDE*

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

is an Ito process. As an equation, it has the form

$$dR(t) = -\beta R(t)dt + d(\alpha t + \sigma W(t)),$$

that is it is a linear equation $dR(t) = -\beta R(t)dt + dX(t)$, forced by the Brownian motion with drift $X(t) = \beta t + \sigma W(t)$. From the Ito formula,

$$d(e^{\beta t}R(t)) = \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t) = e^{\beta t}dX(t).$$

The solution is

$$e^{\beta t}R(t) = R(0) + \int_0^t e^{\beta t}dX(t).$$

Cox-Ingersoll-Ross interest rate model, Example 4.4.11

The solution of the *non linear SDE*

$$dR(t) = (\alpha - \beta R(t))dt + \sqrt{R(t)}\sigma dW(t)$$

is an Ito process. We can write

$$dR(t) = -\beta R(t)dt + (\alpha dt + \sqrt{R(t)}\sigma dW(t)) = -\beta R(t)dt + dY(t),$$

which suggests to compute

$$d(e^{\beta t}R(t)) = \beta e^{\beta t}R(t) + e^{\beta t}dR(t)$$
$$= e^{\beta t}\alpha dt + e^{\beta t}\sigma\sqrt{R(t)}dW(t).$$

The expected value is computable. Same for the second moment.

Black-Scholes-Merton equation, §4.5 I

Portfolio value $X(t)$

Stock value $dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$

Share $\Delta(t)$

Share value $\Delta(t)S(t)$

Differential portfolio value $dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$

We have

$$\begin{aligned} d(\mathrm{e}^{-rt}S(t)) &= (\alpha - r)\mathrm{e}^{-rt}S(t)dt + \sigma\mathrm{e}^{-rt}S(t)dW(t) \\ d(\mathrm{e}^{-rt}X(t)) &= \Delta(t)d(\mathrm{e}^{-rt}S(t)) \end{aligned}$$

Black-Scholes-Merton equation, §4.5 II

let us assume that the the call at time t is a function of stock value $S(t)$, $c(t, S(t))$ and let us compute the differential of the discounted call $\mathrm{e}^{-rt}c(t,x)$ by the Ito-Doeblin forlula. From

$$\begin{aligned} \frac{\partial}{\partial t}\mathrm{e}^{-rt}c(t,x) &= \mathrm{e}^{-rt}(-rc(t,x) + c_{10}(t,x)) \\ \frac{\partial}{\partial x}\mathrm{e}^{-rt}c(t,x) &= \mathrm{e}^{-rt}c_{01}(t,x) \\ \frac{\partial \mathrm{e}^{-rt}c(t,x)}{\partial x} &= \mathrm{e}^{-rt}c_{02}(t,x) \end{aligned}$$

we obtain

$$\begin{aligned} d(\mathrm{e}^{-rt}c(t,S(t))) =& \mathrm{e}^{-rt}(-rc(t,S(t)) + c_{10}(t,S(t)))dt \\ &+ \mathrm{e}^{-rt}c_{01}(t,S(t))dS(t) + \frac{1}{2}\mathrm{e}^{-rt}c_{02}(t,S(t))d[S](t) \end{aligned}$$

Black-Scholes-Merton equation, §4.5 III

and, substituting the differentials

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) \\ d[S](t) &= \sigma^2 S^2(t)dt \end{aligned}$$

we get

$$\begin{aligned} d(\mathrm{e}^{-rt}c(t,S(t))) =& \mathrm{e}^{-rt}(-rc(t,S(t)) + c_{10}(t,S(t)))dt \\ &+ \mathrm{e}^{-rt}c_{01}(t,S(t))(\alpha S(t)dt + \sigma S(t)dW(t)) \\ &+ \frac{1}{2}\mathrm{e}^{-rt}c_{02}(t,S(t))\sigma^2 S^2(t)dt \end{aligned}$$

Now we look for an equation for $c(t,x)$ such that

$$d(\mathrm{e}^{-rt}X(t)) = d(\mathrm{e}^{-rt}c(t,S(t)))$$

We are comparing two Ito process. Forst we equate the martingale terms

$$\mathrm{e}^{-rt}c_{01}(t,S(t))\sigma S(t)dW(t) = \mathrm{e}^{-rt}\Delta(t)\sigma S(t)dW(t).$$

Black-Scholes-Merton equation, §4.5 IV

The equality is true if

$$\Delta(t) = c_{01}(t,S(t)).$$

$$\begin{aligned} &\mathrm{e}^{-rt}c_{01}(t,S(t))(\alpha - r)S(t)dt = \\ &\mathrm{e}^{-rt}(-rc(t,S(t)) + c_{10}(t,S(t)) + c_{01}(t,S(t))\alpha S(t) + c_{02}(t,S(t))\sigma^2 S^2(t))dt \end{aligned}$$

The equality follows if $c(t,x)$ satisfies the *BSM equation*

$$\left(\frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial}{\partial x}\right) c(t,x) = rc(t,x), \quad t \in [0,T], x \geq 0,$$

together with a suitable *boudary condition* e.g.,

$$c(T,x) = (x-K)^+.$$

Multiple Brownian Motion I

Definition (Multiple Brownian motion)

On $(\Omega, \mathcal{F}, \mathrm{P}, (\mathcal{F}(t))_{t\geq 0})$, a *d-dimensional Brownian motion* is a process

$$W(t) = (W_1(t), \dots, W_d(t)), \quad t \geq 0$$

where

1. each W_i is a BM for $\mathcal{F}(t)$, $t \geq 0$,
2. $W_1, \dots, W_d$ are independent,
3. $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, $s < t$.

Theorem (Quadratic variation of the d-dim BM)

$$W(t) \otimes W(t) = \int_0^t W(s) \otimes dW(s) + \int_0^t dW(s) \otimes W(s) + I_d t,$$

Multiple Brownian Motion II

where: $a \otimes b = ab^T = [a_i b_j]_{i=1,\dots,d; j=1,\dots,d}$, $I = [\delta_{i,j}]_{i=1,\dots,d; j=1,\dots,d}$.

Example $d = 2$ and proof

In the case $d = 2$, $W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$ and the terms expand as follows.

$$W(t) \otimes W(t) = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix} \otimes \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix} = \begin{bmatrix} W_1^2(t) & W_1(t)W_2(t) \\ W_2(t)W_1(t) & W_2^2(t). \end{bmatrix}$$

$$W(s) \otimes dW(t) = \begin{bmatrix} W_1(s) \\ W_2(s) \end{bmatrix} \otimes \begin{bmatrix} dW_1(s) \\ dW_2(s) \end{bmatrix} = \begin{bmatrix} W_1(s)dW_1(s) & W_1(s)dW_2(s) \\ W_2(s)dW_1(s) & W_2(t)dW_2(s) \end{bmatrix}.$$

$$dW(s) \otimes W(t) = \begin{bmatrix} dW_1(s) \\ dW_2(s) \end{bmatrix} \otimes \begin{bmatrix} W_1(s) \\ W_2(s) \end{bmatrix} = \begin{bmatrix} dW_1(s)W_1(s) & dW_1(s)W_2(s) \\ dW_2(s)W_1(s) & dW_2(t)W_2(s) \end{bmatrix}.$$

In particular,

$$W_1(t)W_2(t) = \int_0^t W_1(s)dW_2(s) + \int_0^t W_2(s)dW_1(s)$$

Proof: $b \otimes b - a \otimes a = a \otimes (b - a) + (b - a) \otimes a + (b - a) \otimes (b - a)$.

Multivariate Taylor formula

For $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^2)$ the Taylor approximation of the increment from $(s, x) = (s, x_1, x_2)$ to $(t, y) = (t, y_1, y_2)$ is

$$f(t, y_1, y_2) - f(s, x_1, x_2) =$$
$$f_{100}(s, x_1, x_2)(t - s) + f_{010}(s, x_1, x_2)(y_1 - x_1) + f_{001}(s, x_1, x_2)(y_2 - x_2) +$$
$$\frac{1}{2} f_{020}(s, x_1, x_2)(y_1 - x_1)^2 + f_{011}(s, x_1, x_2)(y_1 - x_1)(y_2 - x_2) + \frac{1}{2} f_{002}(s, x_1, x_2)(y_2 - x_2) +$$
$$R_2(s, t, x, y),$$

or, in vector form,

$$f(t, y) - f(s, x) =$$
$$f_s(s, x)(t - s) + f_x(s, x)(y - x) + \frac{1}{2} f_{xx}(s, x) \bullet (y - x)^{\otimes 2} +$$
$$R_2(s, t, x, y),$$

where f_x is the gradient row vector, f_{xx} is the Hessian matrix, $A \bullet B$ is the scalar product of matrices.

Multivariate Ito process

Definition (Multivariate Ito process)

An *Ito process* is a process of the form

$$X(t) = X(0) + \int_0^t \mathbf{\Delta}(s) dW(s) + \int_0^t \mathbf{\Theta}(s) ds,$$

where X and $\mathbf{\Theta}$ are vectors of the same dimension and $\mathbf{\Delta}$ is a matrix of the proper dimensions.

Theorem (Quadratic variation of X)

$$X(t) \otimes X(t) = \int_0^t X(s) \otimes dX(s) + \int_0^t dX(s) \otimes X(s) + \int_0^t \mathbf{\Delta}(s) \circ \mathbf{\Delta}(s) ds,$$

where $A \circ B = AB^T$ is the matrix whose i, j element is the scalar product of the i row of A and the j row of B.

Multi-dimensional Ito-Doeblin formula

Theorem

If

- $f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^d)$,
- X *is a Ito process with* $dX(t) = \mathbf{\Delta}(t) dW(t) + \mathbf{\Theta}(t) dt$,
- $f_x(t, X(t)) \mathbf{\Delta}(t)$, $t \geq 0$, *is an* L^2 *process,*

then $f(t, X(t))$, $t \geq 0$, *is an Ito process, and*

$$f(t, X(t)) =$$
$$f(0, X(0)) + \int_0^t f_t(s, X(s)) ds + \int_0^t f_x(s, X(s)) dX(s) + \frac{1}{2} \int_0^t f_{xx}(s, X(s)) \bullet d[X](s)$$

Notation: $f_t(t, x) = \frac{\partial}{\partial t} f(t, x)$, $f_x(t, x)$ is the *row* gradient vector $\left[\frac{\partial}{\partial x_1} f(t, x_1, \dots, x_d), \dots, \frac{\partial}{\partial x_d} f(t, x_1, \dots, x_d)\right]$, $f_{xx}(t, x)$ is the Hessian matrix $\left[\frac{\partial^2}{\partial x_i \partial x_j} f(t, x_1, \dots, x_d)\right]_{i=1,\dots,d; j=1,\dots,d}$, $A \bullet B$ is the matrix scalar product.

Ito-Doeblin formula for $d = 2$

- $\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} \Delta_{11}(s) dW_1(s) + \Delta_{12}(s) dW_2(s) \\ \Delta_{21}(s) dW_1(s) + \Delta_{22}(s) dW_2(s) \end{bmatrix} + \begin{bmatrix} \Theta_1(s) ds \\ \Theta_2(s) ds \end{bmatrix}$.
- $d[X](t) = \mathbf{\Delta}(t) \circ \mathbf{\Delta}(t) dt =$ $\begin{bmatrix} \Delta_{11}(t)\Delta_{11}(t) + \Delta_{12}(t)\Delta_{12}(t) & \Delta_{11}(t)\Delta_{21}(t) + \Delta_{12}(t)\Delta_{22}(t) \\ \text{simmetric!} & \Delta_{21}(t)\Delta_{21}(t) + \Delta_{22}(t)\Delta_{22}(t) \end{bmatrix} dt$
- $f_t(t, x) = f_{100}(t, x_1, x_2)$.
- $f_x(t, x) = \begin{bmatrix} f_{010}(t, x_1, x_2) & f_{001}(t, x_1, x_2) \end{bmatrix}$.bf
- $f_{xx}(t, x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(t, x_1, x_2) & \frac{\partial^2}{\partial x_1 \partial x_2} f(t, x_1, x_2) \\ \text{simmetric!} & \frac{\partial f}{\partial x_2}(t, x_1, x_2) \end{bmatrix}$
- $\frac{1}{2} f_{xx}(t, x) \bullet \mathbf{\Delta}(t) \circ \mathbf{\Delta}(t) =$ $\frac{1}{2} \frac{\partial f}{\partial x_1}(t, x_1, x_2) \left(\Delta_{11}^2(t) + \Delta_{12}^2(t)\right) + \frac{\partial^2}{\partial x_1 \partial x_2} f(t, x_1, x_2) \Delta_{11}(t) \Delta_{21}(t) + \Delta_{12}(t) \Delta_{22}(t) + \frac{1}{2} \frac{\partial f}{\partial x_2}(t, x_1, x_2) \left(\Delta_{21}^2(t) + \Delta_{22}^2(t)\right)$

Applications

Example The product of two Ito processes $X_1(t) X_2(t)$ is the function $f(x) = x_1 x_2$ of the vector Ito process $(t) = (X_1(t), X_2(t)$. Note that the Hessian has zero diagonal elements while the other two elements are 1. If the Ito process depend on a d-dimensional BM,

$$d(X_1(t) X_2(t)) = X_1(t) dX_2(t) + dX_1(t) X_2(t) + \sum_{j=1}^d \Delta_{1j}(t) \Delta_{2j}(t) dt.$$

Theorem (Lévy theorem $d = 1$)

A continuous L^2 martingale whose quadratic variation is t is a Brownian motion

Theorem (Lévy theorem for generic dimension d)

A continuous L^2 multivariate martingale whose quadratic variation is It is a multivariate Brownian motion.

Proof Compute the moment generating function with the Ito formula.