Stochastic Processes 2014

2. Wiener Process

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Revised June 9, 2014

Plan

1. Poisson Process (Formal construction)
2. Wiener Process (Formal construction)
3. Infinitely divisible distributions (Lévy-Khinchin formula)
4. Lévy processes (Generalities)
5. Stochastic analysis of Lévy processes (Generalities)

References

Billingsley Patrick Billingsley, *Probability and measure*, second ed., Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986

JP Jean Jacod and Philip Protter, *Probability essentials*, second ed., Universitext, Springer-Verlag, Berlin, 2003

McKean Henry P. McKean, *Stochastic integrals*, AMS Chelsea Publishing, Providence, RI, 2005, Reprint of the 1969 edition, with errata

Pintacuda Nicolò Pintacuda, *Probabilità*, Decibel-Zanichelli, Padova, 1994

R R Core Team, *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria, 2014

Royden H. L. Royden, *Real analysis*, third ed., Macmillan Publishing Company, New York, 1988

Sage W. A. Stein et al., *Sage Mathematics Software (Version 5.9)*, The Sage Development Team, 2013, `http://www.sagemath.org`

Williams David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991.

1. Recap: Standard Gaussian distribution

On the multivariate Gaussian distribution cf. [JP].

Construction

- $X \sim \mathrm{N}(0,1)$ if, and only if, its density is $x \mapsto (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$.
- $\mathbf{X} = (X_1, \ldots, x_n) \sim \mathrm{N}_n(0, I)$ if, and only if, its components are independent and $\mathrm{N}(0,1)$. Equivalently, the density is $\mathbf{x} \mapsto (2\pi)^{-n/2}\mathrm{e}^{-\|\mathbf{x}\|^2/2}$.
- $\mathbf{X} \sim \mathrm{N}_n(0, I)$ if, and only if, the characteristic function is $\mathbf{t} \mapsto \mathrm{e}^{\|\mathbf{t}\|/2}$.
- If $\mathbf{X} \sim \mathrm{N}_n(0, I)$ and $U = [\mathbf{u}_1 \cdots \mathbf{u}_n]$ is unitary i.e. $U^T U = I$, then $U\mathbf{X} \sim \mathrm{N}_n(0, I)$.

E1

Proofs. The properties of the characteristic function shall be discussed later.

2. Recap: General Gaussian Distribution

Affine transformations

1. Let $\mathbf{X} \sim \mathrm{N}_n(0, I)$, $\boldsymbol{\mu} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $\Gamma = AA^T$, $\mathbf{Y} = \boldsymbol{\mu} + A\mathbf{X}$. Γ is symmetric and positive definite. The distribution of $\mathbf{Y}$ depends on Γ. Such a distribution is called $\mathrm{N}_m(\boldsymbol{\mu}, \Gamma)$.
2. If $\mathbf{Y} \sim \mathrm{N}_n(\boldsymbol{\mu}, \Gamma)$, $\mathbf{b} \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times n}$, then $\mathbf{b} + A\mathbf{Y} \sim \mathrm{N}_m(\mathbf{b} + B\boldsymbol{\mu}, B\Gamma B^T)$.
3. Given any $\boldsymbol{\mu} \in \mathbb{R}^n$ and any symmetric positive definite $\Gamma \in S_n^+$, the distribution $\mathrm{N}(\boldsymbol{\mu}, \Gamma)$ exists.
4. The characteristic function of $\mathbf{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \Gamma)$ is $\mathbf{t} \mapsto \exp\left(\boldsymbol{\mu}^T\mathbf{t} + \mathbf{t}^T\Gamma\mathbf{t}/2\right)$.
5. $\mathbf{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \Gamma)$ if, and only if, all linear combinations $\sum_j a_j Y_j$ are univariate Gaussian $\mathrm{N}(0, \mathbf{a}^T\Gamma\mathbf{a})$.

E2

Proofs.

3. Recap: Gaussian Distribution Conditioning

Density and conditioning

1. If $\mathbf{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \Gamma)$ and $\det \Gamma \neq 0$, the Y has density $\mathbf{y} \mapsto (2\pi)^{-n/2} (\det \Gamma)^{-1/2} \exp\left(-\mathbf{y}^T \Gamma^{-1/2} \mathbf{y}^t\right)$.
2. If $\mathbf{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \Gamma)$, the blocks $\mathbf{Y}_I = (Y_i : i \in I)$ and $\mathbf{Y}_J = (Y_j : j \in J)$ are independent if, and only if, $\Gamma_{ij} = 0$ for all $i \in I$, $j \in J$, i.e. independence and uncorrelation are equivalent.
3. If $(\mathbf{Y}_1, \mathbf{Y}_2) \sim \mathrm{N}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}\right)$, then the conditional distribution of $\mathbf{Y}_1$ given $\mathbf{Y}_2$ is

$$\mathrm{N}_{n_1}\left(\boldsymbol{\mu}_1 + L_{12}(\mathbf{Y}_2 - \boldsymbol{\mu}_2), \Gamma_{11} - L_{12}\Gamma_{21}\right), \quad L_{12}\Gamma_{22} = \Gamma_{12}.$$

E3

Proofs.

4. Recap: Hilbert spaces

Scalar product

Let $(x, y) \mapsto \langle x, y \rangle$ be a scalar product on a vector space $V \ni x, y$, i.e. a symmetric bilinear mapping such that $\|x\|^2 = \langle x, x \rangle > 0$ unless $x = 0$.

1. $x \mapsto \|x\| = \sqrt{\langle x, x \rangle}$ is a norm. If this norm is complete, then $(V, \langle \cdot, \cdot \rangle)$ is called an *Hilbert space*. E.g. $L^2[0, 1]$, $L^2(\mathbb{P})$.
2. Let $(\phi_n)_{n \in \mathbb{N}}$ be an *orthonormal* sequence in the Hilbert space. Then the series $\sum_{n=1}^{\infty} a_n \phi_n$ is convergent if, and only if $\sum_{n=1}^{\infty} a_n^2 < +\infty$. The limit f satisfies $\|f\|^2 = \sum_{n=1}^{\infty} a_n^2$ and $\langle f, \phi_n \rangle = a_n$, $n \in \mathbb{N}$. $(\phi_n)_{n \in \mathbb{N}}$ be an orthonormal *basis* if $\langle f, \phi_n \rangle = 0$, $n \in \mathbb{N}$ implies $f = 0$.
3. Given two vector spaces V, W, each one having a scalar product, a mapping $A \colon V \to W$ is called an *isometry* if $\langle Ax, Ay \rangle_W = \langle x, y \rangle_V$, $x, y \in V$. If A is an isometry, then A is linear.

E4

Prove item (3).

5. Wiener process = Brownian motion

The *filtration* of the *basis* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ on which a stochastic process is defined is frequently larger than the filtration generated by the process.

Definition

W is a *Brownian motion* for $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ if W is a continuous process, $W\colon \Omega \to C([0, +\infty[)$, such that

1. W is adapted, i.e. W_t is $\mathcal{F}_t$-measurable, $t > 0$,
2. W starts from 0, i.e. $W_0 = 0$ a.s.,
3. the increments are Gaussian, precisely $(W_t - W_s) \sim \mathrm{N}(0, t - s)$, $0 \leq s < t$,
4. the increments are independent from the past history, i.e. $(W_t - W_s)$ is independent of $\mathcal{F}_s$, $0 \leq s < t$.

6. Properties of W

Theorem

1. *The random variables* $W_{t_1}, (W_{t_2} - W_{t_1}), \dots (W_{t_n} - W_{t_{n-1}})$ *are independent if* $0 < t_1 < \cdots < t_n$.
2. *The vector* $(W_{t_1}, \dots, W_{t_n})$, $0 < t_1 < \cdots < t_n$, *has density*

$$p(y_1, \dots, y_n) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} (t_j - t_{j-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}} \right).$$

3. W *is Markov with kernel* $k(x, y) = \frac{1}{2\pi\sqrt{t-s}} \exp\left(-\frac{1}{2}\frac{(y-x)^2}{t-s}\right)$.
4. W, $(W_t^2 - t)_{t>0}$, *and* $\left(\exp\left(aW_t - \frac{a^2}{2} t\right)\right)_{t\geq 0}$, $a \in \mathbb{R}$, *are martingales.*

E5

Proofs.

7. Wiener integral: first step

Simple integrand

1. For each left-continuous time interval $]a, b] \in \mathbb{R}_+$, define $\int_a^b dW_t = \int (a < t \leq b)\ dW_t = W_b - W_a$, so that $\int_a^b dW_t \sim \mathrm{N}(0, t - s)$. If $]s_1, s_2]$ and $]t_1, t_2]$ are left-continuous intervals, then $\mathbb{E}\left(\left(\int (s_1 < t \leq s_2)\ dW_t\right)\left(\int (t_1 < t \leq t_2)\ dW_t\right)\right) = \int (s_1 < t \leq s_2)(t_1 < t \leq t_2)\ dt$.
2. On each left-continuous simple function $f(t) = \sum_{j=1}^n f_{j-1}(t_{j-1} < t \leq t_j)$, define $\int f(t)\ dW_t = \sum_{j=1}^n f_{j-1} \int_{t_{j-1}}^{t_j} dW_t \sim \mathrm{N}(0, \int |f(t)|^2\ dt)$. If f and g are left-continuous simple functions, then $\mathbb{E}\left(\left(\int f(t)\ dW_t\right)\left(\int g(t)\ dW_t\right)\right) = \int f(t)g(t)\ dt$
3. The mapping $f \mapsto \int f(t)\ dW_t$ is linear.

E6

Proofs.

8. Wiener integral of L^2 functions

General integrand

1. Given any $f \in L^2([0,+\infty[)$, there exists a sequence of left-continuous simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ in $L^2([0,+\infty[)$, i.e. $\lim_{n\to\infty} \int |f(t) - f_n(t)|^2 \ dt = 0$.
2. $\int f(t)\ dW_t = \lim_{n\to\infty} \int f_n(t)\ dW_t$ exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $\lim_{n\to\infty} \mathbb{E}\left(\left(\int f(t)\ dW_t - \int f_n(t)\ dW_t\right)\right) = 0$, and the limit does not depend on the approximating sequence.
3. $\int f(t)\ dW_t \sim \mathrm{N}\left(0, \int |f(t)|^2 \ dt\right)$; for each $f, g \in L^2([0,+\infty[)$, the *isometric property* $\mathbb{E}\left(\left(\int f(t)\ dW_t\right)\left(\int g(t)\ dW_t\right)\right) = \int f(t)g(t)\ dt$ holds.
4. The mapping $f \mapsto \int f(t)\ dW_t$ is linear.

We shall discuss later the existence of a *continuous* stochastic process $f \bullet W$ such that $(f \bullet W)_t = \int_0^t f(s)\ dW_s$.

E7

Proofs.

9. Calculus of the Wiener integral

Properties

1. If $f \in L^2([0,+\infty[) \cap C([0,+\infty[)$, then

$$\lim \sum_j f(t_{j_1})(W_{t_j} - W_{t_{j-1}}) = \int f(t)\, dW_t,$$

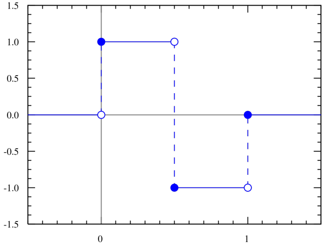
   where the limit is taken along any sequence of partition such that $\max(t_j - t_{j_1}) \to 0$ and $t_n \to \infty$.

2. If $f \in L^2([0,+\infty[) \cap C^1([0,+\infty[)$, then

$$\int_s^t f(u)\, dW_u = f(t)W_t - f(s)W_s - \int_s^t f'(u)W_u\, du.$$

3. If $(\phi_n)_{n\in\mathbb{Z}_+}$ is an orthonormal basis of $L^2([0,1])$ and $a_n(t) = \int_0^t \phi_n(s)\, ds$ fot $0 \le t \le 1$, then there exists a *Gaussian white noise* $Z_0, Z_1, Z_2, \dots$ such that $W_t = \sum_n a_n(t) Z_n$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, namely $Z_n = \int_0^1 \phi_n(t)\, dW_t$.

10. Haar functions

1. Haar functions are $h_0 = 1$, $h_{1,1} =$  ,

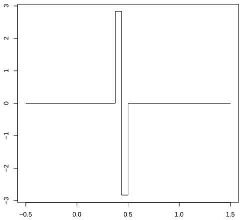
   $h_{1,n}(t) = 2^{(n-1)/2} h_{1,1}(2^{-n+1}t)$, $h_{j,n}(t) = h_{j,1}(t - 2^{-j+1})$, that is for $n \geq 1$ and $j = 1, \ldots, 2^{n-1}$,

$$h_{j,n}(t) = \begin{cases} 2^{(n-1)/2} & \text{if } \frac{2(j-1)}{2^n} \leq t < \frac{2j-1}{2^n}, \\ -2^{(n-1)/2} & \text{if } \frac{2j-1}{2^n} \leq t < \frac{2j}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

2. The Haar function $h_{j,n}$ is zero outside the interval $\left[\frac{2(j-1)}{2^n}, \frac{2j}{2^n}\right[$, whose length is 2^{-n+1}, and where the value is $\pm 2^{(n-1)/2}$.

11. Haar basis

```
left.limit.haar <-
   function(j,n){L1 <- c(2*(j-1)/2^n,sqrt(2^(n-1)))
                 L2 <- c((2*j-1)/2^n,-sqrt(2^(n-1)))
                 L3 <- c(2*j/2^n,0)
                 Ls <- c(L1,L2,L3); Ls
                }
Ls <- left.limit.haar(4,4)
x <- c(-.5,Ls[1],Ls[3],Ls[5],1.5)
y <- c(0,Ls[2],Ls[4],Ls[6],0)
plot(x,y,type="s",xlab="",ylab="")
```



1. The system $(h_0, h_{j,n} \colon n \in \mathbb{N}, j = 1, 2, \dots 2^{n-1})$ is an orthonormal basis of $L^2[0,1]$.
2. The primitives of the Haar functions are the *Shauder functions* and are *tent functions*:

$$\int_0^t h_{j,n}(u)\, du = \begin{cases} 2^{(n-1)/2}\left(t - \frac{2(j-1)}{2^n}\right) & \text{if } \frac{2(j-1)}{2^n} \le t < \frac{2j-1}{2^n}, \\ -2^{(n-1)/2}\left(t - \frac{2j}{2^n}\right) & \text{if } \frac{2j-1}{2^n} \le t < \frac{2j}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

12. Existence of the Wiener process

Theorem

Let $Z_0, Z_{j,n}$, $n = 1, 2, \dots$ *and* $j = 1, \dots, 2^{n-1}$ *be IID* $\mathrm{N}(0,1)$. *Define for each* $n = 1, 2, \dots$ *the continuous Gaussian process* $W^N = F_0 Z_0 + \sum_{n \le N} F_{j,n} Z_{j,n}$.

1. *The sequence* $(W^N)_{N \in \mathbb{N}}$ *converges uniformly almost surely to a continuous process* W.
2. *For each* t *the sequence of random variables* $(W^N(t))_{N \in \mathbb{N}}$ *converges to* $W(t)$ *almost surely and in* $L^2(\mathbb{P})$, *and* $W(t) \sim (0, t)$.
3. *The continuous process is Gaussian, i.e. all finite dimensional distribution are multivariate Gaussian.*
4. *Increments over two disjoint intervals of* W *are uncorrelated, hence independent.*
5. W *is a Wiener process for the filtration it generates.*