# 2.E WIENER PROCESS

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1. **Exercise.** *The random variables*

$$W_{t_1}, (W_{t_2} - W_{t_1}), \dots, (W_{t_n} - W_{t_{n-1}}) \tag{1}$$

*are independent if* $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$.

*Proof.* Note that we can assume $0 < t_1 < cdots < t_n$. Proceed by induction on $n$:

$$\begin{aligned} &\mathrm{E}\left(f_1(W_{t_1})f_2(W_{t_2}-W_{t_1})\cdots f_n(W_{t_n}-W_{t_{n-1}})\right) = \\ &\qquad \mathrm{E}\left(f_1(W_{t_1})f_2(W_{t_2}-W_{t_1})\cdots \mathrm{E}\left(f_n(W_{t_n}-W_{t_{n-1}})\,\middle|\mathcal{F}_{t_{n_1}}\right)\right) = \\ &\qquad\qquad \mathrm{E}\left(f_1(W_{t_1})f_2(W_{t_2}-W_{t_1})\cdots\right)\mathrm{E}\left(f_n(W_{t_n}-W_{t_{n-1}})\right). \end{aligned} \tag{2}$$

□

2. **Exercise.** *The random vector* $(W_{t_1}, \dots, W_{t_n})$, $0 < t_1 < \cdots < t_n$ *is Gaussian with covariance* $\mathrm{Cov}\left(W_{t_i}, W_{t_j}\right) = \min(t_i, t_j)$. *The density exists.*

*Proof.* The increments (1) are independent and $\mathrm{N}(0, t_j - t_{j-1})$, $j = 1, \dots, n$, $t_0 = 0$, with joint density

$$\begin{aligned} p(x_1,\dots,x_n) &= \prod_{j=1}^n (2\pi)^{-1/2}(t_j - t_{j-1})^{-1/2} \exp\left(-\frac{1}{2(t_j - t_{j-1})}x_j^2\right) \\ &= (2\pi)^{-n/2}\left(\prod_{j=1}^n (t_j - t_{j-1})\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{x_j^2}{t_j - t_{j-1}}\right) \end{aligned} \tag{3}$$

The transformation from the increments $\boldsymbol{x}$ to the values $\boldsymbol{y}$, $A\colon \boldsymbol{x} \mapsto \boldsymbol{y}$ is

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & & & \end{bmatrix} \boldsymbol{x} \qquad \boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ \vdots & & & \end{bmatrix} \boldsymbol{y} \tag{4}$$

The determinant of $A$ is 1, hence

$$\begin{aligned} p(y_1,\dots,y_n) &= (2\pi)^{-n/2}\left(\prod_{j=1}^n (t_j - t_{j-1})\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right), \quad t_0 = 0 \\ &= (2\pi)^{-n/2}\left(\prod_{j=1}^n (t_j - t_{j-1})\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{y_j^2 + y_{j-1}^2 - 2y_{j-1}y_j}{t_j - t_{j-1}}\right). \end{aligned} \tag{5}$$

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The quadratic form

$$(6)\quad \boldsymbol{y} \mapsto \sum_{j=1}^{n} \frac{y_j^2 + y_{j-1}^2 - 2y_{j-1}y_j}{t_j - t_{j-1}} =$$

$$\frac{1}{t_1} y_1^2 + \sum_{j=2}^{n-1} \left( \frac{1}{t_j - t_{j-1}} + \frac{1}{t_{j+1} - t_j} \right) y_j^2 + \frac{1}{t_n - t_{n-1}} y_n^2 - 2 \sum_{j=1}^{n} \frac{1}{t_j - t_{j-1}} y_{j-1} y_j$$

is better described in terms of the covariance matrix $\Gamma = \operatorname{Cov}(W_{t_1}, \dots, W_{t_n})$

(7)

$$\Gamma_{ij} = \operatorname{Cov}\left(W_{t_i}, W_{t_j}\right) = \operatorname{Cov}\left(W_{\min(t_i,t_j)}, W_{\max(t_i,t_j)}\right) = \operatorname{Cov}\left(W_{\min(t_i,t_j)}, W_{\min(t_i,t_j)}\right) = \min(t_i, t_j),$$

and

$$(8)\quad \Gamma = A \operatorname{diag}\left(\sqrt{t_j - t_{j-1}} \colon j = 1, \dots, n\right) \left(A \operatorname{diag}\left(\sqrt{t_j - t_{j-1}} \colon j = 1, \dots, n\right)\right)^T = A \operatorname{diag}\left((t_j - t_{j-1}) \colon j = 1, \dots, n\right) A^T$$

and

$$(9)\qquad \Gamma^{-1} = (A^{-1})^T \operatorname{diag}\left(\frac{1}{t_j - t_{j-1}} \colon j = 1, \dots, n\right) (A^{-1})$$

□

3. **Exercise.** *The distribution of $W_t$ given $W_1 = 0$, $t < 1$ is equal to the distribution of the* Brownian bridge $W_t - tW_1$*.*

*Proof.* The joint distribution of $(W_t, W_1)$ is $\mathrm{N}_2\left(0, \begin{bmatrix} t & t \\ t & 1 \end{bmatrix}\right)$, with $\det \begin{bmatrix} t & t \\ t & 1 \end{bmatrix} = t - t^2$ and $\begin{bmatrix} t & t \\ t & 1 \end{bmatrix}^{-1} = \frac{1}{t-t^2} \begin{bmatrix} 1 & -t \\ -t & t \end{bmatrix}$. The joint density is

$$(10)\qquad p_{t,1}(y_1, y_2) = (2\pi)^{-1}(t - t^2)^{-1/2} \exp\left(-\frac{1}{2(t - t^2)}\left(y_1^2 - 2ty_1y_2 + ty_2^2\right)\right)$$

and the conditional density is

$$(11)\quad p_{W_t|W_1}(y_1|y_2) = \frac{(2\pi)^{-1}(t - t^2)^{-1/2} \exp\left(-\frac{1}{2(t-t^2)}\left(y_1^2 - 2ty_1y_2 + ty_2^2\right)\right)}{(2\pi)^{-1/2} \exp\left(-\frac{y_2^2}{2}\right)} =$$

$$(2\pi)^{-1/2}(t - t^2)^{-1/2} \exp\left(-\frac{1}{2(t - t^2)}\left(y_1^2 - 2ty_1y_2 + ty_2^2 - (t - t^2)y_2^2\right)\right) =$$

$$(2\pi)^{-1/2}(t - t^2)^{-1/2} \exp\left(-\frac{1}{2(t - t^2)}\left(y_1^2 - 2ty_1y_2 + t^2y_2^2\right)\right).$$

In particular, because of the continuity, we can define

$$(12)\qquad p_{W_t|W_1}(y_1|0) = (2\pi)^{-1/2}(t - t^2)^{-1/2} \exp\left(-\frac{1}{2(t - t^2)}\left(y_1^2\right)\right).$$

We have $\operatorname{Var}(W_t - tW_1) = \operatorname{Var}(W_t) + t^2 \operatorname{Var}(W_1) - 2t \operatorname{Cov}(W_t, W_1) = t + t^2 - 2t^2 = t - t^2$. □

4. **Exercise.** *The Wiener process is a Markov process an a martingale.*

*Proof.* For all $\phi$ such that $\phi \circ W_t$ is integrable and $s < t$

(13)
$$\mathrm{E}\left(\phi(W_t)\,|\mathcal{F}_s\right) = \mathrm{E}\left(\phi(W_t - W_s + W_s)\,|\mathcal{F}_s\right) = \int \phi(x+W_s)\left((2\pi)^{-1/2}(t-s)^{-1/2}\mathrm{e}^{-\frac{x^2}{2(t-s)}}\right)\,dx = \int \phi(y)\left((2\pi)^{-1/2}(t-s)^{-1/2}\mathrm{e}^{-\frac{(y-W_s)^2}{2(t-s)}}\right)\,dy = \int \phi(y)k(W_s,y)\,dy,$$

that is the conditional distribution of $W_t$ given $\mathcal{F}_s$ is $\mathrm{N}(W_s, t-s)$. In particular, if $\phi(y) = y$, then $\mathrm{E}\left(W_t\,|\mathcal{F}_s\right) = W_s$. □

5. **Exercise.** *Define* $\delta f(x) = xf(x) - f'(x)$ *and* $\delta^n 1 = H_n(x)$. *For* $Z \sim \mathrm{N}(0,1)$, *the random variables* $H_n(Z)$ *are orthogonal.*

*Proof.* As each $H_n$ is a monic polynomial of degree $n$, and

(14)
$$\mathrm{E}\left(f'(Z)g(Z)\right) = (2\pi)^{-1/2}\int f'(z)g(z)\mathrm{e}^{-z^2/2}\,dz = -(2\pi)^{-1/2}\int f(z)\frac{d}{dz}\left(g(z)\mathrm{e}^{-z^2/2}\right)\,dz = -(2\pi)^{-1/2}\int f(z)\left(\frac{d}{dz}g(z)\mathrm{e}^{-z^2/2} - zg(z)\mathrm{e}^{-z^2/2}\right)\,dz = \mathrm{E}\left(f(Z)\delta g(Z)\right),$$

for $m < n$

(15)
$$\mathrm{E}\left(H_m(Z)H_n(Z)\right) = \mathrm{E}\left(H_m(Z)\delta^n 1\right) = \mathrm{E}\left(d^n H_m(Z)\right) = 0.$$

□

6. **Exercise.** *Let* $\phi\colon \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ *and* $\phi \in C^{2,1}$. *Under integrability conditions, if* $t \in \mathbb{R}_+$ *and* $\frac{1}{2}\frac{\partial^2}{\partial y^2}\phi(y,t) + \frac{\partial}{\partial t}\phi(y,t) = 0$, *then* $M = \phi(W,\cdot)$ *is a martingale.*

*Proof.* As $k(x,y;s,t) = \frac{1}{\sqrt{t-s}}f((y-x)/\sqrt{t-s})$ with $f(z) = (2\pi)^{-1/2}\mathrm{e}^{-z^2/2}$, in the limit $t \downarrow s$

(16)
$$\lim_{t\downarrow s} k(x,y;s,t) = \lim_{u\to\infty} u\mathrm{e}^{-u^2(y-x)^2/2} = \begin{cases} 0 & \text{if } y \neq x \\ +\infty & \text{if } y = x \end{cases} = k(x,y,s,s).$$

We have $f'(z) = -zf(z)$, $f''(z) = (z^2-1)f(z)$ and $\frac{d}{dt}(t-s)^{-1/2} = -\frac{1}{2}(t-s)^{-3/2}$. It follows that

(17)
$$\frac{\partial^2}{\partial y^2}k(x,y;s,t) = \frac{\partial^2}{\partial y^2}(t-s)^{-1/2}f((t-s)^{-1/2}(y-x)) = (t-s)^{-3/2}f''((t-s)^{-1/2}(y-x)) = (t-s)^{-3/2}((t-s)^{-1}(y-x)^2-1)f((t-s)^{-1/2}(y-x))$$

and

(18)
$$\frac{\partial}{\partial t}k(x,y;s,t) = \frac{\partial}{\partial t}(t-s)^{-1/2}f((t-s)^{-1/2}(y-x)) = -\frac{1}{2}(t-s)^{-3/2}f((t-s)^{-1/2}(y-x)) + (t-s)^{-1/2}f'((t-s)^{-1/2}(y-x))\left(-\frac{1}{2}(t-s)^{-3/2}(y-x)\right) = -\frac{1}{2}(t-s)^{-3/2}f((t-s)^{-1/2}(y-x)) - \frac{1}{2}(t-s)^{-2}(y-x)f'((t-s)^{-1/2}(y-x)) = -\frac{1}{2}(t-s)^{-3/2}f((t-s)^{-1/2}(y-x)) + \frac{1}{2}(t-s)^{-5/2}(y-x)^2f((t-s)^{-1/2}(y-x)) = \frac{1}{2}(t-s)^{-3/2}\left((t-s)^{-1}(y-x)^2-1\right)f((t-s)^{-1/2}(y-x)),$$

hence

(19)
$$\frac{1}{2}\frac{\partial^2}{\partial y^2}k(x,y;s,t) = \frac{\partial}{\partial t}k(x,y;s,t).$$

We want to show that $\mathrm{E}\left(\phi(W_t,t)\,|\mathcal{F}_s\right) = \phi(W_s,s)$, that is $\int \phi(y,t)k(W_s,y;s,t)\;dy = \phi(W_s,s)$, which, in turn, is implied by $\int \phi(y,t)k(x,y;s,t)\;dy = \phi(x,s)$, $x \in \mathbb{R}$. The function $t \mapsto \int \phi(y,t)k(x,y;s,t)\;dy$ is defined for $t > s$ and with derivative equal to

$$
\begin{aligned}
(20)\quad \frac{\partial}{\partial t}\int \phi(y,t)k(x,y;s,t)\;dy &= \int \frac{\partial}{\partial t}\phi(y,t)k(x,y;s,t)\;dy = \\
&\int \left(\frac{\partial}{\partial t}\phi(y,t)k(x,y;s,t) + \phi(y,t)\frac{\partial}{\partial t}k(x,y;s,t)\right)\;dy = \\
&\int \left(\frac{\partial}{\partial t}\phi(y,t)k(x,y;s,t) + \frac{1}{2}\phi(y,t)\frac{\partial^2}{\partial y^2}k(x,y;s,t)\right)\;dy = \\
&\int \frac{\partial}{\partial t}\phi(y,t)k(x,y;s,t)\;dy + \frac{1}{2}\int \phi(y,t)\frac{\partial^2}{\partial y^2}k(x,y;s,t)\;dy = \\
&\int \frac{\partial}{\partial t}\phi(y,t)k(x,y;s,t)\;dy + \frac{1}{2}\int \frac{\partial^2}{\partial y^2}\phi(y,t)k(x,y;s,t)\;dy = \\
&\int \frac{\partial}{\partial t}\phi(y,t)k(x,y;s,t)\;dy - \int \phi(y,t)k(x,y;s,t)\;dy = 0.
\end{aligned}
$$

For example,

$$
\left(\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}\right)(y^2 - t) = 0 \tag{21}
$$

$$
\left(\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}\right)\mathrm{e}^{ay - a^2 t/2} = 0 \tag{22}
$$

□

Other proofs are possible, namely

$$
t \mapsto \int \phi(y,t)k(x,y;s,t)\;dy = \int phi(x + (t-s)^{1/2}z, t)(2\pi)^{-1/2}\mathrm{e}^{-z^2/2}, \tag{23}
$$

has zero derivative for $t > s$.

7. **Exercise.** *The series of continuous function on* $[0,1]$

$$
t \mapsto tZ_o + \sum_{n=1}^{\infty}\sum_{j=1}^{2^{n-1}}\left(\int_0^t h_{j,n}(s)\;ds\right)Z_{j,n} = \sum_{n=0}^{\infty} W_t^{(n)}, \tag{24}
$$

*with* $Z_0$, $Z_{j,n}$, $n \in \mathbb{N}$, $j = 1\dots 2^{n-1}$, *are IID* $N(0,1)$, *and* $h_{j.n}$ *is the* $j$*-th Haar function of the* $n$*-th order, is a Wiener process on* $[0,1]$ *for its filtration. See* [1, §2.3]

*Proof.*

(1) The Haar function are orthogonal in $L^2[0,1]$ because different functions of the same order have disjoint supports and functions of different orders are such that the one with lower order is constant on the support of the other. The proof of the completeness use a monotone class argument taking as a $\pi$-class the binary intervals.

(2) $W_t^{(0)} = tZ_0$ has the correct distribution at $t = 1$, i.e. $W_1^{(0)} \sim \mathrm{N}(0,1)$. If $W$ is a Wiener process, then $W_t^{(0)} \sim tW_1$, in general $(t \mapsto W_t^{(0)}) \sim (t \mapsto tW_1)$.

(3) If $n = 1$, $2^{n-1} = 1$, and the Haar function is $h_{1,1}(s) = (0 \leqslant s < 1/2) - (1/2 \leqslant s < 1)$ and the Shauder function is $S_{1,1}(t) = \int_0^t h_{1,1}(s)\;ds = t(0 \leqslant t < 1/2) + (1/2 - t)(1/2 \leqslant t < 1)$. The approximation of order 1 is $W_t^{(1)} = tZ_0 + S_{1,1}(t)Z_{1,1}$. At the points $t = 0, 1/2, 1$ the Gaussian vector is

$$
\begin{bmatrix} W_0^{(1)} \\ W_{1/2}^{(1)} \\ W_1^{(1)} \end{bmatrix} = \begin{bmatrix} 0Z_0 + S_{1,1}(0) \\ \frac{1}{2}Z_0 + S_{1,1}\left(\frac{1}{2}\right) \\ 1Z_0 + S_{1,1}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}Z_0 + \frac{1}{2}Z_{1,1} \\ Z_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} Z_0 \\ Z_{1,1} \end{bmatrix} \tag{25}
$$

with covariance

$$\text{Cov}(W_0^{(1)}, W_{1/2}^{(1)}, W_1^{(1)}) = \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \tag{26}$$

so that $(W_0^{(1)}, W_{1/2}^{(1)}, W_1^{(1)}) \sim (W_0, W_{1/2}, W_1)$.

(4) If $n = 2$, then $2^{n-1} = 2$ and $2^{(n-1)/2} = \sqrt{2}$. The Haar function of order 2 are

$$h_{1,2}(s) = \sqrt{2}(0 \leqslant s < 1/4) - \sqrt{2}(1/4 \leqslant s < 1/2), \tag{27}$$

$$h_{2,2}(s) = \sqrt{2}(1/2 \leqslant s < 3/4) - \sqrt{2}(3/4 \leqslant s < 1), \tag{28}$$

and the Shauder functions are

$$S_{1,2}(t) = \sqrt{2}t(0 \leqslant t < 1/4) + \sqrt{2}(1/2 - t)(1/4 \leqslant t < 1/2), \tag{29}$$

$$S_{2,2}(t) = \sqrt{2}(t - 1/2)(1/2 \leqslant t < 3/4) + \sqrt{2}(1 - t)(3/4 \leqslant t < 1). \tag{30}$$

The approximation of order 2 is

$$W_t^{(2)} = tZ_0 + S_{1,1}(t)Z_{1,1} + S_{1,2}(t)Z_{1,2} + S_{2,2}(t)Z_{2,2}. \tag{31}$$

At the binary points $t = 0, 1/4, 1/2, 3/4, 1$ the Gaussian vector is

$$\begin{bmatrix} W_0^{(2)} \\ W_{1/4}^{(2)} \\ W_{1/2}^{(2)} \\ W_{3/4}^{(2)} \\ W_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ (1/4)Z_0 + (1/4)Z_{1,1} + (\sqrt{2}/4)Z_{1,2} \\ (1/2)Z_0 + (1/2)Z_{1,1} \\ (3/4)Z_0 + (1/4)Z_{1,1} + (\sqrt{2}/4)Z_{2,2} \\ Z_0 \end{bmatrix} = \tag{32}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & \sqrt{2}/4 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 1/4 & 0 & \sqrt{2}/4 \\ 1 & & & \end{bmatrix} \begin{bmatrix} Z_0 \\ Z_{1,1} \\ Z_{1,2} \\ Z_{2,2} \end{bmatrix}$$

with covariance

$$\text{Cov}(W_0^{(2)}, W_{1/4}^{(2)}, W_{1/2}^{(2)}, W_{3/4}^{(2)}, W_1^{(2)}) = \tag{33}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & \sqrt{2}/4 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 1/4 & 0 & \sqrt{2}/4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & \sqrt{2}/4 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 1/4 & 0 & \sqrt{2}/4 \\ 1 & 0 & 0 & 0 \end{bmatrix}^T =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1 \\ 0 & 1/4 & 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 & 3/4 & 3/4 \\ 0 & 1/4 & 1/2 & 3/4 & 1 \end{bmatrix} = \text{Cov}(W_0, W_{1/4}, W_{1/2}, W_{3/4}, W_1).$$

(5) To proceed, we want a better organization of the computations. At order 0 we have a zero at $t = 0$; at order 1 we have zeros at $t = 0, 1$; at order 2 we have zeros at $t = 0, 1/2, 1$. At the points $t = 0, 1/2, 1$ we have $W_t^{(2)} = W_t^{(1)}$. As the new point $1/4, 3/4$ the values of $W_t^{(1)}$ are interpolated values, to which are added the vertex values of the new Shauder functions. The increments on binary points are:

$$(34)\quad \begin{bmatrix} W^{(2)}_{1/4} - W^{(2)}_{0} \\ W^{(2)}_{1/2} - W^{(2)}_{1/4} \\ W^{(2)}_{3/4} - W^{(2)}_{1/2} \\ W^{(2)}_{1} - W^{(2)}_{3/4} \end{bmatrix} =$$

$$\begin{bmatrix} (1/4-0)Z_0 + (S_{1,1}(1/4) - S_{1,1}(0))Z_{1,1} + (S_{1,2}(1/4) - S_{1,2}(0))Z_{1,2} + (S_{2,2}(1/4) - S_{2,2}(0))Z_{2,2} \\ (1/2-1/4)Z_0 + (S_{1,1}(1/2) - S_{1,1}(1/4))Z_{1,1} + (S_{1,2}(1/2) - S_{1,2}(1/4))Z_{1,2} + (S_{2,2}(1/2) - S_{2,2}(1/4))Z_{2,2} \\ (3/4-1/2)Z_0 + (S_{1,1}(3/4) - S_{1,1}(1/2))Z_{1,1} + (S_{1,2}(3/4) - S_{1,2}(1/2))Z_{1,2} + (S_{2,2}(3/4) - S_{2,2}(1/2))Z_{2,2} \\ (1-3/4)Z_0 + (S_{1,1}(1) - S_{1,1}(3/4))Z_{1,1} + (S_{1,2}(1) - S_{1,2}(3/4))Z_{1,2} + (S_{2,2}(1) - S_{2,2}(3/4))Z_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} (1/4)Z_0 + S_{1,1}(1/4)Z_{1,1} + S_{1,2}(1/4)Z_{1,2} \\ (1/4)Z_0 + (S_{1,1}(1/2) - S_{1,1}(1/4))Z_{1,1} - S_{1,2}(1/4)Z_{1,2} \\ (1/4)Z_0 + (S_{1,1}(3/4) - S_{1,1}(1/2))Z_{1,1} + S_{2,2}(3/4)Z_{2,2} \\ (1/4)Z_0 - S_{1,1}(3/4)Z_{1,1} - S_{2,2}(3/4))Z_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} (1/4)Z_0 + (1/2)S_{1,1}(1/2)Z_{1,1} + S_{1,2}(1/4)Z_{1,2} \\ (1/4)Z_0 + (1/2)S_{1,1}(1/2)Z_{1,1} - S_{1,2}(1/4)Z_{1,2} \\ (1/4)Z_0 - (1/2)S_{1,1}(1/2)Z_{1,1} + S_{2,2}(3/4)Z_{2,2} \\ (1/4)Z_0 - (1/2)S_{1,1}(1/2)Z_{1,1} - S_{2,2}(3/4))Z_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} (1/4)Z_0 \\ (1/2)S_{1,1}(1/2)Z_{1,1} \\ S_{1,2}(1/4)Z_{1,2} \\ S_{2,2}(3/4)Z_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} (1/4)Z_0 \\ (1/4)Z_{1,1} \\ (\sqrt{2}/4)Z_{1,2} \\ (\sqrt{2}/4)Z_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & \sqrt{2}/4 & 0 \\ 0 & 0 & 0 & \sqrt{2}/4 \end{bmatrix} \begin{bmatrix} Z_0 \\ Z_{1,1} \\ Z_{1,2} \\ Z_{2,2} \end{bmatrix}$$

and the covariance is

$$(35)\quad \mathrm{Cov}(W^{(2)}_{1/4} - W^{(2)}_{0}, W^{(2)}_{1/2} - W^{(2)}_{1/4}, W^{(2)}_{3/4} - W^{(2)}_{1/2}, W^{(2)}_{1} - W^{(2)}_{3/4}) =$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/16 & 0 & 0 & 0 \\ 0 & 1/16 & 0 & 0 \\ 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}^T =$$

$$\frac{1}{8} \begin{bmatrix} 1/2 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & -1 & 0 \\ 1/2 & -1/2 & 0 & 1 \\ 1/2 & -1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}^T$$

(6) Let us prove the convergence. This proof is due to [2] and uses the Borel-Cantelli lemma [3, §2.7]. First note that $Z \sim \mathrm{N}(0,1)$ implies the following estimate of the queues:

(36)

$$\mathrm{P}\left(|Z| > x\right) = (2\pi)^{-1/2} \int_{|z|>x} \mathrm{e}^{-z^2/2}\, dz = \sqrt{\frac{2}{\pi}} \int_x^\infty \mathrm{e}^{-z^2/2}\, dz \leqslant \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{z}{x} \mathrm{e}^{-z^2/2}\, dz = \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-x^2/2}}{x}.$$

It follows that

(37)
$$\mathrm{P}\left(\max_j |Z_{j,n}| > A(n)\right) = \mathrm{P}\left(\cup_j \{|Z_{j,n}| > A(n)\}\right) \leqslant \sum_j \mathrm{P}\left(|Z_{j,n}| > n\right) = 2^{n-1}\,\mathrm{P}\left(|Z| > A(n)\right) \leqslant 2^{n-1}\sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-A(n)^2/2}}{A(n)}.$$

□

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