Stochastic Processes and Calculus 2016

Wiener Process

Giovanni Pistone

Collegio Carlo Alberto

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References

- On multivariate Gaussian: Handout 1;
- On Martingales: Part B of David Williams, Probability with martingales, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991;
- On Wiener calculus: Ch 4 of Robert S. Liptser and Albert N. Shiryaev, Statistics of random processes. I, expanded ed., Applications of Mathematics (New York), vol. 5, Springer-Verlag, Berlin, 2001, General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability. 1800857;
- On Ornstein-Uhlembeck process: Handout 2.

Recap: Standard Gaussian distribution

On the multivariate Gaussian distribution cf. handout 2.

Definition

- $X \sim \mathrm{N}(0,1)$ if, and only if, its density is $x \mapsto (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$.
- $\boldsymbol{X} = (X_1, \ldots, x_n) \sim \mathrm{N}_n(0, I)$ if, and only if, its components are independent and $\mathrm{N}(0,1)$. Equivalently, the density is $\boldsymbol{x} \mapsto (2\pi)^{-n/2}\mathrm{e}^{-\|\boldsymbol{x}\|^2/2}$.

Properties

- $\boldsymbol{X} \sim \mathrm{N}_n(0, I)$ if, and only if, the characteristic function is $\boldsymbol{t} \mapsto \mathrm{e}^{\|\boldsymbol{t}\|/2}$.
- If $\boldsymbol{X} \sim \mathrm{N}_n(0, I)$ and $U = [\boldsymbol{u}_1 \cdots \boldsymbol{u}_n]$ is unitary i.e. $U^T U = I$, then $U\boldsymbol{X} \sim \mathrm{N}_n(0, I)$.

Recap: General Gaussian Distribution

Definition

Let $\boldsymbol{X} \sim \mathsf{N}_n(0, I)$, $\boldsymbol{\mu} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $\Sigma = AA^T$, $\boldsymbol{Y} = \boldsymbol{\mu} + A\boldsymbol{X}$. Σ is symmetric and positive definite. The distribution of $\boldsymbol{Y}$ depends on $\boldsymbol{\mu}$ and Σ only. Such a distribution is Gaussian with mean $\boldsymbol{\mu}$ and variance Σ, $\mathsf{N}_m(\boldsymbol{\mu}, \Sigma)$.

Properties

1. If $\boldsymbol{Y} \sim \mathsf{N}_n(\boldsymbol{\mu}, \Sigma)$, $\boldsymbol{b} \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times n}$, then $\boldsymbol{b} + A\boldsymbol{Y} \sim \mathsf{N}_m(\boldsymbol{b} + B\boldsymbol{\mu}, B\Sigma B^T)$.
2. Given any $\boldsymbol{\mu} \in \mathbb{R}^n$ and any symmetric positive definite $\Sigma \in S_n^+$, the distribution $\mathsf{N}(\boldsymbol{\mu}, \Sigma)$ exists.
3. The characteristic function of $\boldsymbol{Y} \sim \mathsf{N}(\boldsymbol{\mu}, \Sigma)$ is $\boldsymbol{t} \mapsto \exp\left(\boldsymbol{\mu}^T \boldsymbol{t} + \boldsymbol{t}^T \Sigma \boldsymbol{t}/2\right)$.
4. $\boldsymbol{Y} \sim \mathsf{N}(\boldsymbol{\mu}, \Sigma)$ if, and only if, all linear combinations $\sum_j a_j Y_j$ are univariate Gaussian $\mathsf{N}(0, \boldsymbol{a}^T \Sigma \boldsymbol{a})$.

Recap: Gaussian distribution properties

Density

If, and only if, $\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ and $\det \Sigma \neq 0$, then Y has a density, namely

$$\boldsymbol{y} \mapsto (2\pi)^{-n/2} (\det \Sigma)^{-1/2} \exp\left(-(\boldsymbol{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{y} - \boldsymbol{\mu})\right)$$

The concentration matrix $K = \Sigma^{-1}$ is symmetric and positive definite.

Independence

If $\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$, the blocks $\boldsymbol{Y}_I = (Y_i \colon i \in I)$ and $\boldsymbol{Y}_J = (Y_j \colon j \in J)$ are independent if, and only if, $\Sigma_{ij} = 0$ for all $i \in I$, $j \in J$.

Conditioning

If $(\boldsymbol{Y}_1, \boldsymbol{Y}_2) \sim \mathrm{N}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$, then the conditional distribution of $\boldsymbol{Y}_1$ given $\boldsymbol{Y}_2$ is

$$\mathrm{N}_{n_1}\left(\boldsymbol{\mu}_1 + L_{12}(\boldsymbol{Y}_2 - \boldsymbol{\mu}_2), \Sigma_{11} - L_{12}\Sigma_{21}\right), \quad L_{12}\Sigma_{22} = \Sigma_{12}.$$

Recap: Hilbert spaces

Scalar product

$(x, y) \mapsto \langle x, y \rangle$ is a scalar product on a real vector space $V \ni x, y$, i.e. a symmetric bilinear mapping such that $\|x\|^2 = \langle x, x \rangle > 0$ unless $x = 0$.

1. $x \mapsto \|x\| = \sqrt{\langle x, x \rangle}$ is a norm. If the norm is complete, then $(V, \langle \cdot, \cdot \rangle)$ is called an Hilbert space. E.g., $L^2[0,1]$ and $L^2(\mathbb{P})$.
2. γ is N(0,1), $V = L^2(\gamma)$ with $\langle f, g \rangle_\gamma = \int f(z) g(z)\ \gamma(dz)$ is an Hilbert space.
3. Let $(\phi_n)_{n \in \mathbb{N}}$ be an orthonormal sequence in the Hilbert space. Then the series $\sum_{n=1}^\infty a_n \phi_n$ is convergent if, and only if $\sum_{n=1}^\infty a_n^2 < +\infty$. The limit f satisfies $\|f\|^2 = \sum_{n=1}^\infty a_n^2$ and $\langle f, \phi_n \rangle = a_n$, $n \in \mathbb{N}$. $(\phi_n)_{n \in \mathbb{N}}$ be an orthonormal basis if $\langle f, \phi_n \rangle = 0$, $n \in \mathbb{N}$ implies $f = 0$.
4. Hermite polynomials $H_n(x) = \delta^n 1$ are an ONB of $L^2(\mathbb{R}, \gamma)$.
5. Given two vector spaces V, W, each one having a scalar product, a mapping $A \colon V \to W$ is called an isometry if $\langle Ax, Ay \rangle_W = \langle x, y \rangle_V$, $x, y \in V$. If A is an isometry, then A is linear.

Wiener process = Brownian motion

Note: the filtration of the basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ on which a stochastic process is defined could be larger than the filtration generated by the process.

Definition

W is a Brownian motion for $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t)_{t\geq 0}))$ if W is a continuous process, $W \colon \Omega \to C([0, +\infty[)$, such that

1. W is adapted, i.e. W_t is $\mathcal{F}_t$-measurable, $t > 0$,
2. W starts from 0, i.e. $W_0 = 0$ a.s.,
3. the increments are Gaussian, precisely $(W_t - W_s) \sim \mathrm{N}(0, t - s)$, $0 \leq s < t$,
4. the increments are independent from the past history, i.e. $(W_t - W_s)$ is independent of $\mathcal{F}_s$, $0 \leq s < t$.

Basic properties of W

Theorem

1. *The random variables* $W_{t_1}, (W_{t_2} - W_{t_1}), \dots (W_{t_n} - W_{t_{n-1}})$ *are independent if* $0 < t_1 < \cdots < t_n$.
2. *The vector* $(W_{t_1}, \dots, W_{t_n})$, $0 < t_1 < \cdots < t_n$, *has density*

$$p(y_1, \dots, y_n) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} (t_j - t_{j-1})^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}} \right).$$

3. W *is Markov with kernel* $k(x, y) = \frac{1}{2\pi\sqrt{t-s}} \exp\left(-\frac{1}{2} \frac{(y-x)^2}{t-s} \right)$.
4. W, $(W_t^2 - t)_{t>0}$, *and* $\left(\exp\left(aW_t - \frac{a^2}{2} t \right) \right)_{t \geq 0}$, $a \in \mathbb{R}$, *are martingales*.

Wiener integral: simple integrand

1. For each left-continuous time interval $]a, b] \in \mathbb{R}_+$, define $\int_a^b dW_t = \int (a < t \leq b)\ dW_t = W_b - W_a$, so that $\int_a^b dW_t \sim \mathrm{N}(0, t - s)$. If $]s_1, s_2]$ and $]t_1, t_2]$ are left-continuous intervals, then

$$\mathbb{E}\left(\left(\int (s_1 < t \leq s_2)\ dW_t\right)\left(\int (t_1 < t \leq t_2)\ dW_t\right)\right) = \int (s_1 < t \leq s_2)(t_1 < t \leq t_2)\ dt$$

2. On each left-continuous simple function $f(t) = \sum_{j=1}^n f_{j-1}(t_{j-1} < t \leq t_j)$, define $\int f(t)\ dW_t = \sum_{j=1}^n f_{j-1} \int_{t_{j-1}}^{t_j} dW_t \sim \mathrm{N}(0, \int |f(t)|^2\ dt)$. If f and g are left-continuous simple functions, then

$$\mathbb{E}\left(\left(\int f(t)\ dW_t\right)\left(\int g(t)\ dW_t\right)\right) = \int f(t)g(t)\ dt$$

3. The mapping $f \mapsto \int f(t)\ dW_t$ is linear.

Wiener integral of L^2 functions

General integrand

1. Given any $f \in L^2([0,+\infty[)$, there exists a sequence of left-continuous simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_n = f$ in $L^2([0,+\infty[)$, i.e. $\lim_{n\to\infty} \int |f(t) - f_n(t)|^2 \; dt = 0$.
2. $\int f(t)\; dW_t = \lim_{n\to\infty} \int f_n(t)\; dW_t$ exists in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, i.e. $\lim_{n\to\infty} \mathbb{E}\left(\left(\int f(t)\; dW_t - \int f_n(t)\; dW_t\right)\right) = 0$, and the limit does not depend on the approximating sequence.
3. $\int f(t)\; dW_t \sim \mathrm{N}\left(0, \int |f(t)|^2 \; dt\right)$; for each $f, g \in L^2([0,+\infty[)$, the isometric property $\mathbb{E}\left(\left(\int f(t)\; dW_t\right)\left(\int g(t)\; dW_t\right)\right) = \int f(t)g(t)\; dt$ holds.
4. The mapping $f \mapsto \int f(t)\; dW_t$ is linear.
5. $\left(\int_0^t f(s)\; dW_s\right)_{t\geq 0}$ has independent increments.
6. There exists a continuous stochastic process $f \bullet W$ such that $(f \bullet W)_t = \int_0^t f(s)\; dW_s$ (From Doob's maximal inequality).

Calculus of the Wiener integral

Properties

1. If $f \in L^2([0,+\infty[) \cap C([0,+\infty[)$, then

$$\lim \sum_j f(t_{j_1})(W_{t_j} - W_{t_{j-1}}) = \int f(t)\ dW_t,$$

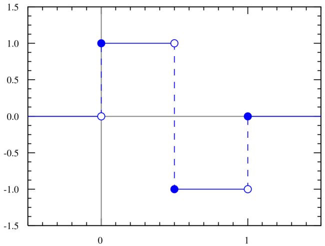
where the limit is taken along any sequence of partition such that $\max(t_j - t_{j_1}) \to 0$ and $t_n \to \infty$.

2. If $f \in L^2([0,+\infty[) \cap C^1([0,+\infty[)$, then

$$\int_s^t f(u)\ dW_u = f(t)W_t - f(s)W_s - \int_s^t f'(u)W_u\ du.$$

3. If $(\phi_n)_{n\in\mathbb{Z}_+}$ is an orthonormal basis of $L^2([0,1])$ and $a_n(t) = \int_0^t \phi_n(s)\ ds$ for $0 \le t \le 1$, then there exists a Gaussian white noise $Z_0, Z_1, Z_2, \ldots$ such that $W_t = \sum_n a_n(t)Z_n$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, namely $Z_n = \int_0^1 \phi_n(t)\ dW_t$.

Haar functions

1. Haar functions are $h_0 = 1$, $h_{1,1} =$ 

,

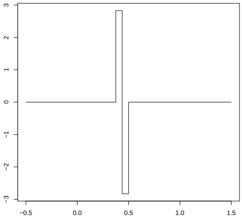
   $h_{1,n}(t) = 2^{(n-1)/2} h_{1,1}(2^{-n+1}t)$, $h_{j,n}(t) = h_{j,1}(t - 2^{-j+1})$, that is for $n \geq 1$ and $j = 1, \ldots, 2^{n-1}$,

$$h_{j,n}(t) = \begin{cases} 2^{(n-1)/2} & \text{if } \frac{2(j-1)}{2^n} \leq t < \frac{2j-1}{2^n}, \\ -2^{(n-1)/2} & \text{if } \frac{2j-1}{2^n} \leq t < \frac{2j}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

2. The Haar function $h_{j,n}$ is zero outside the interval $\left[\frac{2(j-1)}{2^n}, \frac{2j}{2^n}\right[$, whose length is 2^{-n+1}, and where the value is $\pm 2^{(n-1)/2}$.

Haar basis

```
left.limit.haar <-
   function(j,n){L1 <- c(2*(j-1)/2^n,sqrt(2^(n-1)))
                 L2 <- c((2*j-1)/2^n,-sqrt(2^(n-1)))
                 L3 <- c(2*j/2^n,0)
                 Ls <- c(L1,L2,L3); Ls
                }
Ls <- left.limit.haar(4,4)
x <- c(-.5,Ls[1],Ls[3],Ls[5],1.5)
y <- c(0,Ls[2],Ls[4],Ls[6],0)
plot(x,y,type="s",xlab="",ylab="")
```



1. The system $\left(h_0, h_{j,n} \colon n \in \mathbb{N}, j = 1, 2, \ldots 2^{n-1}\right)$ is an orthonormal basis of $L^2[0,1]$.
2. The primitives of the Haar functions are the Shauder functions and are tent functions:

$$\int_0^t h_{j,n}(u)\,du = \begin{cases} 2^{(n-1)/2}\left(t - \frac{2(j-1)}{2^n}\right) & \text{if } \frac{2(j-1)}{2^n} \le t < \frac{2j-1}{2^n}, \\ -2^{(n-1)/2}\left(t - \frac{2j}{2^n}\right) & \text{if } \frac{2j-1}{2^n} \le t < \frac{2j}{2^n}, \\ 0 & \text{otherwise.} \end{cases}$$

Existence of the Wiener process

On the construction of the Wiener process see [KS]

Theorem

Let $Z_0, Z_{j,n}$, $n = 1, 2, \ldots$ and $j = 1, \ldots, 2^{n-1}$ be IID $\mathrm{N}(0,1)$. Define for each $n = 1, 2, \ldots$ the continuous Gaussian process $W^N = F_0 Z_0 + \sum_{n \leq N} F_{j,n} Z_{j,n}$.

1. *The sequence $(W^N)_{N \in \mathbb{N}}$ converges uniformly almost surely to a continuous process W.*
2. *For each t the sequence of random variables $(W^N(t))_{N \in \mathbb{N}}$ converges to $W(t)$ almost surely and in $L^2(\mathbb{P})$, and $W(t) \sim (0, t)$.*
3. *The continuous process is Gaussian, i.e. all finite dimensional distribution are multivariate Gaussian.*
4. *Increments over two disjoint intervals of W are uncorrelated, hence independent.*
5. *W is a Wiener process for the filtration it generates.*

Orstein-Uhlembeck process

Definition

Given a Wiener process W and constant $\theta, \sigma > 0$, $\mu \in \mathbb{R}$, the Ornstein-Uhlembeck process is $(X_t^x)_{t \geq 0}$ unique strong solution of

$$dX_t^x = \theta(\mu - X_t^x)dt + \sigma dW_t, \quad X_0^x = x$$

1. We focus on the standard case $dX_t^x = -X_t dt + \sqrt{2}W_t$, $X_0^x = 0$, precisely

$$X_t^x = x - \int_0^t X_u \; du + \sqrt{2}W_t, \quad t \geq 0$$

2. The unique strong solution is

$$\boxed{X_t^x = \mathrm{e}^{-t}x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)} \; dW_s}$$

where the stochastic integral is a Wiener integral.

Proof

1. $\mathrm{e}^t X_t^x + \mathrm{e}^t \int_0^t X_u \, du = \mathrm{e}^t x + \sqrt{2}\mathrm{e}^t W_t$
2. $\frac{d}{dt}\left(\mathrm{e}^t \int_0^t X_u \, du\right) = \mathrm{e}^t x + \sqrt{2}\mathrm{e}^t W_t$
3. $\mathrm{e}^t \int_0^t X_u \, du = (\mathrm{e}^t - 1)x + \sqrt{2}\int_0^t \mathrm{e}^s W_s \, ds$
4. $\int_0^t X_u \, du = (1 - \mathrm{e}^{-t})x + \sqrt{2}\mathrm{e}^{-t}\int_0^t \mathrm{e}^s W_s \, ds$
5.

$$\begin{aligned} X_t &= \mathrm{e}^{-t}x - \sqrt{2}\mathrm{e}^{-t}\int_0^t \mathrm{e}^s W_s \, ds + \sqrt{2}\mathrm{e}^{-t}\mathrm{e}^t W_t \\ &= \mathrm{e}^{-t}x + \sqrt{2}\mathrm{e}^{-t}\left(\mathrm{e}^t W_t - \int_0^t \mathrm{e}^s W_s \, ds\right) \\ &= \mathrm{e}^{-t}x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)} \, dW_s \end{aligned}$$

Distribution of the OU

- Each X_t^x is Gaussian with mean $\mathbb{E}(X_t^x) = \mathrm{e}^{-t}x$ and variance

$$\operatorname{Var}(X_t^x) = 2\,\mathbb{E}\left(\left(\int_0^t \mathrm{e}^{-(t-s)}\, dW_s\right)^2\right) = \int_0^t 2\mathrm{e}^{-2(t-s)}\, ds = \mathrm{e}^{-2(t-s)}\Big|_{s=0}^{s=t} = 1 - \mathrm{e}^{-2t}$$

- For $t > s$ the covariance is

$$\begin{aligned}
\operatorname{Cov}(X_t^x, X_s^x) &= 2\,\mathbb{E}\left(\left(\int_0^t \mathrm{e}^{-(t-u)}\, dW_u\right)\left(\int_0^s \mathrm{e}^{-(s-u)}\, dW_u\right)\right) \\
&= 2\int_0^s \mathrm{e}^{-(t-u)}\mathrm{e}^{-(s-u)}\, du \\
&= \mathrm{e}^{-s-t}\, \mathrm{e}^{2u}\Big|_{u=0}^{u=s} \\
&= \mathrm{e}^{-(t-s)} - \mathrm{e}^{-(t+s)} = \mathrm{e}^{-2(t-s)}(1 - \mathrm{e}^{-2s})
\end{aligned}$$

- OU $(X_t^x)_{t\geq 0}$ is a continuous adapted Gaussian process. Note that $\sigma(X_s^x\colon s \leq t) = \sigma(W_s^x\colon s \leq t)$.

OU is Markov I

- If $s < t$

$$\begin{bmatrix} X_s^x \\ X_t^x \end{bmatrix} \sim \mathrm{N}_2 \left(\begin{bmatrix} \mathrm{e}^{-s} \\ \mathrm{e}^{-t} \end{bmatrix} x, \begin{bmatrix} 1 - \mathrm{e}^{-2s} & \mathrm{e}^{-(t-s)}\mathrm{e}^{-2s} \\ \mathrm{e}^{-(t-s)}\mathrm{e}^{-2s} & 1 - \mathrm{e}^{-2t} \end{bmatrix} \right)$$

  hence

$$X_t^x | (X_s^x = y) \sim \mathrm{N}_1 \left(\mathrm{e}^{-(t-s)y}, 1 - \mathrm{e}^{-2(t-s)} \right)$$

  so that

$$\mathbb{E}\left(f(X_t^x) | (X_S^x = y) \right) = \int f(\mathrm{e}^{-(t-s)} y + \sqrt{1 - \mathrm{e}^{-2(t-s)}} z) \gamma(dz)$$

- We have $X_t^x \sim \mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}} Z$, hence we define the family of operators P_t, $t \geq 0$,

$$(P_t f)(x) = \mathbb{E}\left(f(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}} Z) \right) =$$
$$\int f(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}} z) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2} z^2} \, dz$$

OU is Markov II

- Markov property:

$$
\begin{aligned}
&\mathbb{E}\left(f(X_t^x)|\mathcal{F}_s\right)\\
&= \mathbb{E}\left(f(\mathrm{e}^{-t}x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)}\, dW_s)\Big|\mathcal{F}_s\right)\\
&= \mathbb{E}\left(f(\mathrm{e}^{-t}x + \sqrt{2}\int_0^s \mathrm{e}^{-(t-s)}\, dW_s + \sqrt{2}\int_s^t \mathrm{e}^{-(t-s)}\, dW_s)\Big|\mathcal{F}_s\right)\\
&= \int f(\mathrm{e}^{-t}x + \sqrt{2}\int_0^s \mathrm{e}^{-(t-s)}\, dW_s + \sqrt{1-\mathrm{e}^{-2(t-s)}z})\,\gamma(dz)\\
&= \int f(\mathrm{e}^{-(t-s)}X_s^x + \sqrt{1-\mathrm{e}^{-2(t-s)}z})\,\gamma(dz)\\
&= (P_{t-s}f)(X_s^x)
\end{aligned}
$$

- $(P_t)_{t\geq 0}$ is a semigroup, $P_s \circ P_{t-s} = P_t$ if $s \leq t$, of contraction operators on $L^2(\gamma)$