STOCHASTIC CALCULUS 2013
PART III

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Assigment

Read [2, Ch 3]. All the random variables in the following are defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P}(\cdot))$. Choose two among the exercises discussed below. If you need hints, check the the first set of notes of the year 2011-2012. Deadline: coming week.

1. Normal distribution

Note that the definition [2, Definition 2.2.11] does not cover all cases of interest, i.e. multivariate and/or degenerate cases. See e.g. [1, Ch 16]. Some basic facts are reviewed in the following **Exercise 1**.

(1) Let $Y, X_1, \ldots, X_n$ be a gaussian vector. Compute $\hat{a}_1, \ldots, \hat{a}_n$ such that

$$\mathrm{Cov}\left(Y - \sum_j \hat{a}_j X_j, X_i\right) = 0$$

for all $i = 1, \ldots, n$.

(2) Same notations. Compute the distribution of $\sum_j \hat{a}_j X_j$ and the distribution of $Z = Y - \sum_j \hat{a}_j X_j$.

(3) Same notations. Show that

$$\mathrm{E}\left(Y \,|X_1, \ldots, X_n\right) = \mathrm{E}\left(Y\right) + \sum_j \hat{a}_j (X_j - \mathrm{E}\left(X_j\right)).$$

(4) Same notations. Compute $\mathrm{E}\left(\phi(Y) \,|X_1, \ldots, X_n\right)$.

2. Brownian motion

The brownian motion W is a centered gaussian stochastic process, Hence all joint distributions depend on the covariance

$$\mathrm{Cov}\left(W_s, W_t\right) = \min(s,t), \quad s,t \geq 0.$$

Use the results in Sec. 1 to solve the following **Exercise 2**.

(1) Compute the joint finite dimensional distributions and the joint finite dimensional densities of W.

(2) Given the times $t_1 < t_2 < t_3$, compute the distribution of each W_{t_i}, $i = 1, 2, 3$, given the other two W_{t_i}, W_{t_k}, $i, k \neq j$.

3. Brownian martingales

A brownian martingale is a martingale M which is an adapted function of W. The quadratic variation of a martingale is the limit of the sum of squared increments along the filter of time partitions. The following **Exercise 3** is a warmup before Stochastic Calculus.

(1) Use the law of large numbers to show that the quadratic variation of $(W_t)_t$ is the deterministic process equal to t for all times.

(2) The discrete time process $Y_n = \sum_{k=1}^n W_{k-1}(W_k - W_{k-1})$ is a martingale.

(3) $W_t^2 - t$, $t \geq 0$ is a martingale.

(4) $\exp\left(aW_t - \frac{a^2}{2}t\right)$, $t \geq 0$, is a martingale.

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4. Applications

Two interesting exercises are **Exercise 3.5** and **Exercise 3.6** on [2, p. 118].

References

1. Jean Jacod and Philip Protter, *Probability essentials*, second ed., Universitext, Springer-Verlag, Berlin, 2003. MR MR1956867 (2003m:60002)
2. Steven E. Shreve, *Stochastic calculus for finance. II*, Springer Finance, Springer-Verlag, New York, 2004, Continuous-time models. MR 2057928 (2005c:91001)

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