STOCHASTIC CALCULUS 2013
PART II

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1. Weakly assigment

Read [2, Ch 2-3]. All the random variables in the following are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot))$. Choose two among the exercises discussed below. Deadline: coming week.

2. Conditional Expectation

Theorem 1 (Fubini)**.** *Let X, Y, be independent random variables. Assume $f(X,Y)$ is integrable and define the partial expectation*

$$x \mapsto \widehat{f}(x) = \mathrm{E}\left(f(x,Y)\right).$$

Then

$$\mathrm{E}\left(f(x,Y)\right) = \mathrm{E}\left(\widehat{f}(X)\right).$$

Theorem 2. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. For each $X \in L^1$ there exists a unique $\widetilde{X}$ such that*

1. *$\widetilde{X}$ is $\mathcal{G}$-measurable, and*
2. *$\mathrm{E}\left(YX\right) = \mathrm{E}\left(Y\widetilde{X}\right)$ if Y is bounded and $\mathcal{G}$-measurable.*

The random variable $\widetilde{X}$ is called the conditional expectation of X given $\mathcal{G}$ and it is denoted by $\mathrm{E}\left(X\,|\mathcal{G}\right)$.

Theorem 3. *Let X, Y independent random variables and let $\mathcal{G} = \sigma(X)$. Define the partial expectation*

$$\widehat{f}(x) = \mathrm{E}\left(f(x,Y)\right).$$

Then

$$\mathrm{E}\left(f(X,Y)\,|\mathcal{G}\right) = \widehat{f}(X).$$

Proof. **Exercise 1a** Hint: 1. prove that $f(x,Y)$ is integrable; 2. check the definition of conditional expectation for $\widehat{f}(X)$ by observing that $g(x)\,\mathrm{E}\left(f(x,Y)\right) = \mathrm{E}\left(g(x)f(x,Y)\right)$. □

Theorem 4. *Let Q be a positive density and $\mathrm{E}_Q\left[X\right] = \mathrm{E}\left(QX\right)$. Then*

$$\mathrm{E}_Q\left[X|\mathcal{G}\right] = \frac{\mathrm{E}\left(QX\,|\mathcal{G}\right)}{\mathrm{E}\left(Q\,|\mathcal{G}\right)}.$$

Proof. **Exercise 1b** □

3. Multivariate Hermite polynomials

Define

$$\begin{aligned}\delta f(x) &= xf(x) - f'(x)\\ &= -e^{x^2/2}\frac{d}{dx}\left(f(x)e^{-x^2/2}\right).\end{aligned}$$

If $Z \sim \mathcal{N}(0,1)$, then

$$\mathrm{E}\left(g(Z)\delta f(Z)\right) = \mathrm{E}\left(dg(Z)f(Z)\right),$$

i.e. δ is the transpose of the derivative w.r.t. the standard Gaussian measure. Moreover, $d\delta - \delta d = \mathrm{id}$.

Definition 1. *The Hermite polynomials are*

$$\begin{aligned}H_0 &= 1,\\ H_n(x) &= \delta^n 1, n > 0.\end{aligned}$$

The first Hermite polynomials are

$$\begin{aligned}H_1(x) &= x,\\ H_2(x) &= x^2 - 1,\\ H_3(x) &= x^3 - 3x,\\ H_4(x) &= x^4 - 6x^2 + 3, \dots\end{aligned}$$

Theorem 5.

1. *Let $Z \sim \mathrm{N}(0,1)$. The random variables $H_n(Z)$ are orthogonal, $\mathrm{E}\left(H_n(Z)H_m(Z)\right) = 0$ if $n \neq m$, and $\mathrm{E}\left(H_n(Z)^2\right) = n!$.*
2. *$dH_n = nH_{n-1}$,*
3. *$H_{n+1} = xH_n - nH_{n-1}$.*

Proof.

1. Let $m \leq n$. The conclusion follows from
$$\mathrm{E}\left(H_n(Z)H_m(Z)\right) = \mathrm{E}\left(d^n H_m(Z)\right)$$
by observing that $H_m(z) = z^m + \cdots$.
2. By induction: $dH_1(x) = 1$, and

$$\begin{aligned}dH_n(x) &= d\delta H_{n-1}(x) = \delta d H_{n-1}(x) + H_{n-1}(x)\\ &= (n-1)\delta H_{n-2}(x) + H_{n-1}(x) = nH_{n-1}(x)\end{aligned}$$

□

Theorem 6. *Let $Z_1, \dots, Z_k$ be IID $\mathrm{N}(0,1)$ and define the multivariate standard normal $\boldsymbol{Z} = (Z_1, \dots, Z_k)$. The random variables*

$$\frac{1}{\boldsymbol{n}!}H_{\boldsymbol{n}}(\boldsymbol{Z}) = \frac{1}{n_1!}H_{n_1}(Z_1)\cdots\frac{1}{n_k!}H_{n_k}(Z_k),$$

where $\boldsymbol{n} = (n_1, \dots, n_k)$ is a multi-index and $\boldsymbol{n}! = n_1!\cdots n_k!$, are an orthonormal basis of $L^2(\sigma(\boldsymbol{Z}))$.

Proof. See [1, V-1.6]. □

4. Martingales and Markov Processes

A discrete time real stochastic process $Y_0, Y_1, \dots$ adapted to the filtation $\mathcal{F}_t$, $t = 0, 1, \dots$ is a *martingale* if it is integrable and

$$\mathrm{E}\left(Y_t\,|\mathcal{F}_s\right) = Y_s, \quad \text{if } s \leq t.$$

Theorem 7. *Let $(Y_t, \mathcal{F}_t \colon t = 1, 2, \dots)$ be a square-integrable martingale.*

1. *The process*
$$A_t = \sum_{s \leq t}\mathrm{E}\left(Y_s^2 - Y_{s-1}^2\,|\mathcal{F}_{t-1}\right)$$
is integrable, increasing and predictable*, i.e. A_t is $\mathcal{F}_{t-1}$-measurable.*
2. *$(Y_t^2 - A_t, \mathcal{F}_t \colon t = 1, 2, \dots)$ is a martingale.*

The process A is called the compensator *of Y^2.*

Proof. **Exercise 2**. □

Definition 2. *A discrete time stochastic process $Y_0, Y_1, \dots$ adapted to the filtration $\mathcal{F}_0, \mathcal{F}_1, \dots$ is a* Markov process *if for all $s < t$ and all real function f such that $f(Y_t)$ is integrable, there exists a real function $\widehat{f}_{s,t}$ such that*

$$\mathrm{E}\left(f(Y_t)\,|\mathcal{F}_s\right) = \widehat{f}_{s,t}(Y_s)$$

If

$$\widehat{f}_{t-1,t}(x) = \int f(y)\kappa(x,y)dy$$

the function κ is called kernel *or* transition probability *of the Markov process.*

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5. Gaussian Random Walk

Let $Z_1, Z_2, \dots$ be IID $\mathrm{N}(0,1)$ and define the *Gaussian Random Walk* GRW to be the stochastic process

$$\begin{aligned} W_0 &= 0, \\ W_1 &= Z_1 = W_0 + Z_1, \\ W_2 &= Z_1 + Z_2 = W_1 + Z_2, \\ W_3 &= Z_1 + Z_2 + Z_3 = W_2 + Z_3, \\ &\vdots \end{aligned}$$

Theorem 8. (1) *The GRW is a Gaussian process.*
(2) *The random vector $(W_1, \dots, W_{t-1})$ and the random variable $(W_t - W_{t-1}) = Z_t$ are independent.*
(3) *The GRW is a Markov process with kernel*

$$\kappa(x,y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-x)^2\right).$$

(4) *The GRW is a square-integrable martingale.*
(5) *The compensator of Y^2 is $A_n = n$.*

Proof. **Exercise 3** □

Theorem 9 (Hermite martingales of the GRW)**.** *Chose an Hermite polynomial H_n and define the real stochastic process*

$$\begin{aligned} X_0 &= 0, \\ X_t &= t^{\frac{n}{2}} H_n\left(\frac{W_t}{\sqrt{t}}\right), \quad t = 1, 2, \dots \end{aligned}$$

The process $(X_t)_t$ is a square integrable martingale for the filtration where F_0 is trivial and

$$F_t = \sigma(W_1, \dots, W_t) = \sigma(Z_1, \dots, Z_t), \quad t > 0.$$

Proof. **Exercise 4** Hint. We want to prove

$$\mathrm{E}\left(t^{\frac{n}{2}} H_n\left(\frac{W_t}{\sqrt{t}}\right) | \mathcal{F}_{t-1}\right) = (t-1)^{\frac{n}{2}} H_n\left(\frac{W_{t-1}}{\sqrt{t-1}}\right)$$

that is

$$\mathrm{E}\left(t^{\frac{n}{2}} H_n\left(\frac{w_{t-1} + Z_n}{\sqrt{t}}\right)\right) = (t-1)^{\frac{n}{2}} H_n\left(\frac{w_{t-1}}{\sqrt{t-1}}\right).$$

Try first the case $n = 1, 2, \dots$. Write $\frac{w_{t-1}}{\sqrt{t-1}} = y$ to reduce to

$$\left(\frac{t}{t-1}\right)^{\frac{n}{2}} \mathrm{E}\left(H_n\left(\frac{\sqrt{t-1}y + Z_n}{\sqrt{t}}\right)\right) = H_n(y),$$

and apply to the LHS the definition of Hermite polynomial. □

Exercise 4': It is interesting to compare the beaviour of the different Hermite martingales with a simulation, for example $n = 1, 2, 3, 4$ on $t = 0, 1, \dots, 100$. First obtain a random sample $z_, \dots, z_{100}$, compute the cumulated sums, then the Hermite polynomials, and plot.

References

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2. Steven E. Shreve, *Stochastic calculus for finance. II*, Springer Finance, Springer-Verlag, New York, 2004, Continuous-time models. MR 2057928 (2005c:91001)

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