STOCHASTIC CALCULUS 2013
PART I

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1. Weakly assigment

Read [2, Ch 1-2]. All the random variables in the following are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot))$. Choose two among the exercises discussed below. Deadline: coming week.

2. Expectation

Definition 1. *If X is a nonnegative random variable, $X \in L^+$, its expectation is well defined and equal to*

$$\mathrm{E}(X) = \int_0^{+\infty} \mathbb{P}(X > x)\, dx.$$

The mapping $x \mapsto \mathbb{P}(X > x)$ is nonincreasing, hence the integral is an ordinary integral. If $X \sim B(p)$, then $\mathrm{E}(X) = p$.

Theorem 1.
1. *If $X \in L^+$, then $\mathrm{E}(X) = 0$ if, and only if $X = 0$ a.s.*
2. *If $X, Y \in L^+$, then $X + Y \in L^+$ and $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$.*
3. *If $X, Y \in L^+$ and $X \leq Y$, then $\mathrm{E}(X) \leq \mathrm{E}(Y)$, with equality if, and only if, $X = Y$ a.s.*
4. *If $X_n \uparrow X$, $n \to \infty$, then $\mathrm{E}(X_n) \uparrow \mathrm{E}(X)$.*

Proof. **Exercise 1** □

Definition 2. *If X is a real random variable, and both $\mathrm{E}(X^+)$ and $\mathrm{E}(X^-)$ are finite, then $X \in L^1$ and its expected value is*

$$\mathrm{E}(X) = \mathrm{E}\left(X^+\right) - \mathrm{E}\left(X^-\right).$$

Theorem 2. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. For each $X \in L^1$ there exists a unique $\widetilde{X}$ such that*
1. *$\widetilde{X}$ is $\mathcal{G}$-measurable, and*
2. *$\mathrm{E}(YX) = \mathrm{E}\left(Y\widetilde{X}\right)$ if Y is bounded and $\mathcal{G}$-measurable.*

The random variable $\widetilde{X}$ is called the conditional expectation of X given $\mathcal{G}$ and it is denoted by $\mathrm{E}(X|\mathcal{G})$.

3. Bregman divergence

Let f be (strictly) convex and C^1 on the open interval $I =]a, b[$, $a \in \{-\infty\} \cup \mathbb{R}$, $b > a$ and in $\mathbb{R} \cup \{+\infty\}$. Define the *Bregman divergence* between x and y in I as

$$d(y; x) = f(y) - f(x) - f'(x)(y - x).$$

The *divergence* is nonnegative and it is zero if, and only if, $x = y$. The mapping $y \mapsto d(y; x)$ is convex with derivative

$$\frac{d}{dy} d(y; x) = f'(y) - f'(x).$$

The mapping $x \mapsto d(y; x)$ is not convex in general. Check as **Exercise 2a** that

$$d(y; x) - d(y; z) = d(z; x) + (f'(z) - f'(x))(y - z) \tag{1}$$

Definition 3. *If X and Y are random variables with values in I, the random variable $d(Y; X)$ is nonnegative and we can define their* Bregman divergence *by*

$$D(y; x) = \mathrm{E}(d(Y; X)).$$

The divergence is nonnegative, possibly $+\infty$, and it is zero if, and only if, $X = Y$ a.s.

If $f(x) = x^2$, then $I = \mathbb{R}$ and

$$\begin{aligned} d(y; x) &= y^2 - x^2 - 2x(y - x) \\ &= (y - x)^2 \end{aligned}$$

and

$$D(Y; X) = \mathrm{E}\left((Y - X)^2\right).$$

If $f(x) = x \ln x - x$ with $I =]0, +\infty[$, then

$$\begin{aligned} d(y; x) &= y \ln y - y - (x \ln x - x) - \ln x (y - x) \\ &= y(\ln y - \ln x) + x - y \\ &= y \ln\left(\frac{y}{x}\right) - y + x. \end{aligned}$$

If Q and P are positive densities, then

$$\begin{aligned} D(Q; P) &= \mathrm{E}\left(Q \ln\left(\frac{Q}{P}\right) - Q + P\right) \\ &= \mathrm{E}\left(Q \ln\left(\frac{Q}{P}\right)\right) - 1 + 1 \\ &= \mathrm{E}\left(Q \ln\left(\frac{Q}{P}\right)\right) = \mathrm{E}_Q\left[\ln\left(\frac{Q}{P}\right)\right]. \end{aligned}$$

If $P = 1$, then $D(Q; 1) = \mathrm{E}(Q \ln(Q))$ is called the *entropy* of Q. Check the keyword *minimal entropy measure* on Google, e.g. `http://www.math.ethz.ch/~mschweiz/Files/MEMM-eqf.pdf`. For other examples of Bregman divergence, see `http://en.wikipedia.org/wiki/Bregman_divergence`.

Theorem 3. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Assume X and Y take values in I and X is $\mathcal{G}$-measurable. If $Y \in L^1$, the conditional expectation $\widetilde{Y} = \mathrm{E}(Y|\mathcal{G})$ takes values in I and*

$$D(Y; X) \geq D(Y; \widetilde{Y}).$$

Viceversa, if $\widetilde{Y}$ is the minimum point of $X \mapsto D(Y; X)$ over all X taking values in I and $\mathcal{G}$-measurable, then $\widehat{Y} = \mathrm{E}(Y|\mathcal{G})$.

Proof. **Exercise2b**
1. If $I =]a, b[$ and a is finite, then $I \subset \{y : y - a > 0\}$. As $Y - a > 0$ a.s., it follows $\mathrm{E}(Y - a|\mathcal{G}) = \widetilde{Y} - a > 0$ a.s.
2. From (1),

$$\begin{aligned} d(Y; X) - d(Y; \widetilde{Y}) &= d(\widetilde{Y}; X) + (f'(\widetilde{Y}) - f'(X))(Y - \widetilde{Y}) \\ &\geq (f'(\widetilde{Y}) - f'(X))(Y - \widetilde{Y}). \end{aligned}$$

   If $(f'(\widetilde{Y}) - f'(X))$ were bounded, the expected value of the inequality would be zero. If not, multiply the inequality by $(-n < f'(\widetilde{Y}) - f'(X) < n)$.
3. If $D(Y; X) = d(Y; \widetilde{Y})$ and $(f'(\widetilde{Y}) - f'(X))$ were bounded, then $D(X; \widetilde{Y}) = 0$.

□

4. Hermite polynomials

Define

$$\begin{aligned} \delta f(x) &= x f(x) - f'(x) \\ &= -e^{x^2/2} \frac{d}{dx}\left(f(x) e^{-x^2/2}\right). \end{aligned}$$

If $Z \sim \mathcal{N}(0, 1)$, then

$$\mathrm{E}(g(Z)\delta f(Z)) = \mathrm{E}(dg(Z) f(Z)),$$

i.e. δ is the transpose of the derivative w.r.t. the standard Gaussian measure. Moreover, $d\delta - \delta d = \mathrm{id}$.

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Definition 4. *The Hermite polynomials are*

$$H_0 = 1,$$
$$H_n(x) = \delta^n 1, n > 0.$$

The first Hermite polynomials are

$$H_1(x) = x,$$
$$H_2(x) = x^2 - 1,$$
$$H_3(x) = x^3 - 3x,$$
$$H_4(x) = x^4 - 6x^2 + 3, \ldots$$

Theorem 4. (1) *Let $Z \sim \mathrm{N}(01)$. The random variables $H_n(Z)$ are orthogonal, $\mathrm{E}(H_n(Z)H_m(Z)) = 0$ if $n \neq m$, and $\mathrm{E}\left(H_n(Z)^2\right) = n!$.*
(2) *$dH_n = nH_{n-1}$,*
(3) *$H_{n+1} = xH_n - nH_{n-1}$.*

Proof. **Exercise 3**. There are various references, e.g. [1, 230–233] □

Theorem 5. *$(H_n(Z)/\sqrt{n!})$, $n = 0, 1, 2\ldots$, is an orthonormal basis of $L^2(\sigma(Z))$.*

A proof is in the reference above. We skip it, but this example is of interest for us:

$$\exp\left(tZ - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(Z).$$

4.1. **Gaussian quadrature.**

Theorem 6. (1) *Each Hermite polynomial H_n, $n \geq 1$, has n distinct real roots.*
(2) *The roots of H_{n+1} are separated by the roots of H_n, $n \geq 1$.*

Let f be a polynomial in one variable with real coefficients. By polynomial division

$$f(x) = q(x)H_n(x) + r(x)$$

where r has degree smaller than H_n and $r(x) = f(x)$ on $\{x \colon H_n(x) = 0\}$. The $n-1$ degree polynomial r is the remainder.

It follows

$$\begin{aligned}\mathrm{E}(f(Z)) &= \mathrm{E}(q(Z)H_n(Z)) + \mathrm{E}(r(Z)) \\ &= \mathrm{E}(q(Z)\,\delta 1^n(Z)) + \mathrm{E}(r(Z)) \\ &= \mathrm{E}(d^n q(Z)) + \mathrm{E}(r(Z)).\end{aligned}$$

Hence

$$\mathrm{E}(f(Z)) = \mathrm{E}(r(Z)) \quad \text{iff} \quad \mathrm{E}(d^n q(Z)) = 0$$

Note that $d^n q(Z) = 0$ if and only if q has degree smaller than n and this is only if f has degree smaller or equal to $2n-1$. The expected value can be zero in many other cases.

If r has degree ness than n, then

$$r(x) = \sum_{i=1}^{n} r(x_i) L_i(x),$$

where $x_1, \ldots, x_n$ are the roots of $H_n(x) = 0$ and

$$L_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

are the Lagrange polynomials. The *quadrature formula* follows:

$$\begin{aligned}\mathrm{E}(f(Z)) &= \mathrm{E}(r(Z)) \\ &= \sum_{i=1}^{n} r(x_i)\,\mathrm{E}(L_i(Z)) \\ &= \sum_{i=1}^{n} r(x_i) w_{n,i}.\end{aligned}$$

The values of the *weights* $w_{n,i}$ are precomputed and available in numerical tables and numerical software. See e.g. the classical tables `http://people.math.sfu.ca/~cbm/aands/` and the relevant R funtions.

Exercise 4 Write in detail the method of quadrature and compute numerically an example.

References

1. Paul Malliavin, *Integration and probability*, Graduate Texts in Mathematics, vol. 157, Springer-Verlag, New York, 1995, With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky. MR MR1335234 (97f:28001a)
2. Steven E. Shreve, *Stochastic calculus for finance. II*, Springer Finance, Springer-Verlag, New York, 2004, Continuous-time models. MR 2057928 (2005c:91001)

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