Stochastic Proceses 2014

1. Poisson Process

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Random variables

Random variable

Let $(S, \mathcal{S})$ and $(R, \mathcal{B})$ be measurable spaces, and $X : S \to R$ a function. The function X is a $(R, \mathcal{B})$-random variable if $X^{-1}(\mathcal{B}) \subset \mathcal{S}$.

If $\mathcal{C} \subset \mathcal{B}$ is a generating set of $\mathcal{B}$, and $X^{-1}(\mathcal{C}) \subset \mathcal{S}$, then X is a random variable. In fact, let $\mathcal{E} = \left\{ B \in \mathcal{B} \middle| X^{-1}(B) \in \mathcal{S} \right\}$. Then $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{B}$, and $\mathcal{C} \subset \mathcal{E}$, then $\mathcal{E} = \mathcal{B}$, see [Williams 3.2.b].
The σ-algebra generated by X is $X^{-1}(\mathcal{B})$. From $\mathcal{C} \subset \mathcal{B}$, we have $X^{-1}(\mathcal{C}) \subset X^{-1}(\mathcal{B})$, hence $\sigma(X^{-1}(\mathcal{C})) \subset X^{-1}(\mathcal{B}) = X^{-1}(\sigma(\mathcal{C}))$. Let $\mathcal{E} = \left\{ B \in \mathcal{B} \middle| X^{-1}(B) \in \sigma(X^{-1}(\mathcal{C})) \right\}$. As $\mathcal{E}$ is a σ-algebra containing $\mathcal{C}$ and contained in in $\mathcal{B}$, then $\mathcal{E} = \mathcal{B}$ and $X^{-1}(\mathcal{B}) = \sigma(X^{-1}(\mathcal{C}))$.

Checking Measurability

Let $(S, \mathcal{S})$ be a measurable space and $X \colon S \to R$. If $\mathcal{C}$ is a family of subsets of R and $X^{-1}(\mathcal{C}) \subset \mathcal{S}$, then

1. X is a random variable in $(S, \sigma(\mathcal{C}))$;
2. $\sigma(X) = X^{-1}(\sigma(\mathcal{C})) = \sigma(X^{-1}(\mathcal{C}))$.

In particular, if $S \subset R$ and $X \colon S \ni x \mapsto x \in R$ is the immersion, then $X^{-1}(\mathcal{C}) = \mathcal{C} \cap S = \{ C \cap S | C \in \mathcal{C} \}$.

2. Stochastic Process

1 Trajectory: definition

Let $\mathcal{I}$ be a real interval, the *set of times*. Let D be a set of real functions $u \colon \mathcal{I} \to \mathbb{R}$, the *set of trajectories*. Given $t \in \mathcal{I}$, the mapping $\Pi_t \colon D \to \mathbb{R}$ defined by $\Pi_t(u) = u(t)$ is the *evaluation* of the trajectory at t. Let $\mathcal{D} = \sigma(\Pi_t \colon t \in \mathcal{I})$ be the σ-algebra generated by the evaluations. The measurable space $(D, \mathcal{D})$ is a *space of trajectories*.

2 Stochastic process: definition

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a space of trajectories $(D, \mathcal{D})$, a *stochastic process* on $(\Omega, \mathcal{F}, \mathbb{P})$ with trajectories in D is a random variable $X : \Omega \to D$, $X^{-1} \colon \mathcal{D} \to \mathcal{F}$. The *distribution* of X is $\mathbb{P} \circ X^{-1}$.

3 Stochastic process: distribution

1. If $t_1, \ldots, t_n \in \mathcal{I}$, then $(X_{t_1}, \ldots, X_{t_n})$ is random variable in $\mathbb{R}^n$ whose distribution is called *finite-dimensional distribution* at $t_1, \ldots, t_n \in \mathcal{I}$.
2. Finite dimensional distributions characterize the distribution of the stochastic process.

E1

Check 3.1–2. Use [Williams.3]: Def 3.1 applies to general range, not just to real valued functions.

The family of events of the form $\left\{\omega \in S \middle| X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\right\}$, for $B_1, \ldots, B_n \in \mathcal{B}$, is a π-system that generates $X^{-1}(\mathcal{D})$. Note that it is enough to take the B's is a π-system for $\mathbb{R}$.

E2

Consider the polynomials $l_0(x) = 1$, $l_1(x) = \sqrt{3}(2x - 1)$, $l_2(x) = \sqrt{5}(6x^2 - 6x + 1)$; check the orthonormality on $[0, 1]$. Define $L_i(t) = \int_0^t l_i(x)dx$. Given Z_0, Z_1, Z_2 iid $N(0, 1)$, define the stochastic process $W^{(2)} = L_0 Z_0 + L_1 Z_1 + L_2 Z_2$. Compute Cov $\left(W_s^{(2)}, W_t^{(2)}\right)$ and the marginal distribution of $(W_s^{(2)}, W_t^{(2)})$

Note that $W^{(2)}$ is a stochastic process whose trajectories are polynomials of degree up to 3, and $\mathbb{E}\left(W_t^{(2)}\right) = \mathbb{E}\left(L_0(t)Z_0 + L_1(t)Z_1 + L_2(t)Z_2\right) = 0.$

$$\int_0^1 l_0(x)l_0(x)\ dx = \int_0^1 1^2\ dx = 1$$
$$\int_0^1 l_1(x)l_1(x)\ dx = \int_0^1 (\sqrt{3}(2x-1))^2\ dx = 3\int_0^1 (4x^2-4x+1)\ dx = 1$$
$$\int_0^1 l_0(x)l_1(x)\ dx = \int_0^1 1(\sqrt{3}(2x-1))\ dx = 0$$
$$\int_0^1 l_2(x)l_2(x)\ dx = \int_0^1 (\sqrt{5}(6x^2-6x+1))^2\ dx = 5\int_0^1 (36x^4-72x^3+48x^2-12x+1)\ dx = 1$$
$$\int_0^1 l_0(x)l_2(x)\ dx = \int_0^1 1(\sqrt{5}(6x^2-6x+1))\ dx = 0$$
$$\int_0^1 l_1(x)l_2(x)\ dx = \int_0^1 ((\sqrt{3}(2x-1)))((\sqrt{5}(6x^2-6x+1)))\ dt = 0$$

$$\mathrm{Cov}\left(W_s^{(2)}, W_t^{(2)}\right) = \mathbb{E}\left((L_0(s)Z_0 + L_1(s)Z_1 + L_2(s)Z_2)(L_0(t)Z_0 + L_1(t)Z_1 + L_2(t)Z_2)\right) =$$
$$\sum_{i,j=0}^{n} L_i(s)L_j(t)\,\mathbb{E}\left(Z_iZ_j\right) = \sum_{i=0}^{n} L_i(s)L_i(t)$$

$$(W_s^{(2)}, W_t^{(2)}) \sim \mathrm{N}\left(0, \begin{bmatrix} \sum_{i=0}^n L_i^2(s) & \sum_{i=0}^n L_i(s)L_i(t) \\ \sum_{i=0}^n L_i(s)L_i(t) & \sum_{i=0}^n L_i^2(t) \end{bmatrix}\right)$$

To be continued in the chapter on Wiener process.

Exponential distribution

The *exponential* distribution $\text{Exp}(\lambda)$ has support $[0, +\infty[$ and it is a model for random times. The parameter $\lambda > 0$ is the *intensity*. *The positive random variable X has exponential distribution with intensity λ ($\lambda > 0$) if its density is $f_X(x) = \lambda \exp(-\lambda x)(x > 0)$.* The *survival function* is 1 if $x < 0$ and for $x \geq 0$ its value is $R_X(x) = \exp(-\lambda x)$. The *distribution function* is 0 if $x < 0$ and for $x \geq 0$ its value is $F_X(x) = 1 - R_X(x) = 1 - \exp(-\lambda x)$. The *quantile function*, $\{x | F_X(x) \geq u\} = \{x | Q_X(u) \leq x\}$, $0 < u < 1$, is $Q_X(u) = -\frac{1}{\lambda} \ln(1-u)$, $\in]0, 1[$. The *expected value* is $\mathbb{E}(X) = \int_0^{+\infty} R_X(x) dx = \int_0^{+\infty} \exp(-\lambda x) dx = \frac{1}{\lambda}$. The *moment generating function* is $M_X(t) = \mathbb{E}(\exp(tX)) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$. The variance is $\text{Var}(X) = \frac{1}{\lambda^2}$. If $X_1, \dots, X_n$ are iid $\text{Exp}(\lambda)$, then $T = X_1 + \cdots + X_n$ is $\Gamma(n, \lambda)$, with density $f_T(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda t}(t > 0)$. [If p is a proposition, the (p) is the truth value, i.e. the indicator function.]

The exponential distribution is memory-less

Easy version

Let $X \sim \text{Exp}(\lambda)$. For each $a, t \geq 0$,

$$\mathbb{P}(X > t + a \mid X > a) = \mathbb{P}(X > t).$$

More sophisticated version

Let $X \sim \text{Exp}(\lambda)$ and let $\mathcal{G}$ be a σ-algebra independent of X. If A and B are positive $\mathcal{G}$-measurable random variables, then for all $t > 0$,

$$\mathbb{E}(B(X > t + A)) = \mathbb{P}(X > t)\,\mathbb{E}(B(X > A)).$$

Proofs: Independence implies (see [Williams 8.3-4])

$$\mathbb{E}(B(X > t + A)) = \mathbb{E}\left(\int_0^{+\infty} B(x > t + A)\ \lambda \mathrm{e}^{-\lambda x} dx\right) = \mathbb{E}\left(B\mathrm{e}^{-\lambda(t+A)}\right) = \mathrm{e}^{-\lambda t}\,\mathbb{E}\left(B\mathrm{e}^{-\lambda A}\right)$$

The *easy version* is $A = a$, $B = 1$, $\mathcal{G}$ trivial. □

E3

Same assumptions: $\mathbb{E}\left(B(X > t + A)\,|\mathcal{G}\right) = B\mathrm{e}^{-\lambda(t+A)}$.

Use [Williams 9.10].

E4

Compute $\mathbb{E}\left((X > t)\,|\sigma(\{X > a\} : a \leq s)\right)$, $s \leq t$.

Let $(S, \mathcal{S}, \mathbb{P})$ the probability space where X is defined and let $X_t = (X \leq t)$ be the stochastic processes that has a unit jump at the random time X. For each time t, the σ algebra $\sigma(X_t)$ is $\{\emptyset, S, \{X \leq t\}, \{X > t\}\}$, with generator $\{X > t\}$. The σ-algebra $\mathcal{F}_s = \sigma(X_a \colon a \leq s)$ has generators $\mathcal{C} = \{\{X > a\} | a \leq s\}$, which is a π-system as $\{X > a_1\} \cap \{X > a_2\} = \{X > \max(a_1, a_2)\}$, see [Williams 1.6]. A bounded random variable B is $\mathcal{F}_s$-measurable if, and only if, $B = g(X)(X \leq s) + c(X > s)$, with g Borel bounded and c constant. In fact the class $\mathcal{H} = \{g(X)(X \leq s) + c(X > s) | g \in \mathcal{B}_b, c \in \mathbb{R}\}$ is a monotone class containing the indicators of $\mathcal{C}$ because $(X > a) = (X > a)(X \leq s) + (X > a)$, see [Williams 3.14]. Let us compute g and c such that $\mathbb{E}\left((X > t)\,|\mathcal{F}_s\right) = g(X)(X \leq s) + c(X > s)$, see [Williams 9.2]. For all $s \leq a$,

$$\mathbb{E}\left((X > a)(X > t)\right) = \mathbb{E}\left((X > a)(g(X)(X \leq s) + c(X > s))\right), \quad a \leq s,$$

hence

$$\exp\left(-\lambda t\right) = \mathbb{E}\left((a < X \leq s) g(X)\right) + c \exp\left(-\lambda s\right), \quad a \leq s.$$

If $a = s$, $\exp\left(-\lambda t\right) = c \exp\left(-\lambda s\right)$, then $c = \exp\left(-\lambda(t - s)\right)$. It follows

$$\mathbb{E}\left((a < X \leq s) g(X)\right) = \int_s^s g(x)\ \lambda \mathrm{e}^{-\lambda x}\ dx = 0, \quad s \leq a$$

so that $g = 0$ and $\mathbb{E}\left((X > t)\,|\mathcal{F}_s\right) = \exp\left(-\lambda(t - s)\right)(X > s)$.

E5

Compute $\mathbb{E}(X_t - X_s \,|\mathcal{F}_s)$, $s \le t$.

From the previous computation,

$$\mathbb{E}(X_t - X_s \,|\mathcal{F}_s) = 1 - \mathbb{E}((X > t)\,|\mathcal{F}_s) - X_s = 1 - \exp(-\lambda(t-s))\,(X > s) - X_s = \\ 1 - \exp(-\lambda(t-s))\,(1 - X_s) - X_s = (\exp(-\lambda(t-s)) - 1)(X_s - 1) \ge 0,$$

then $(X_t, \mathcal{F}_t)_{t \ge 0}$ is a *sub-martingale*, see [Williams 10.3].

E6

Show that $A = \lambda \int_0^{\cdot} (X_u - 1)\, du$ is an absolutely continuous *compensator* of $X_\cdot$, that is it is a process with absolutely continuous trajectories and such that $M = X_\cdot - A$ is a martingale [CD1].

Having checked that it is correct to exchange integration with conditional expectation,

$$\mathbb{E}(A_t - A_s \,|\mathcal{F}_s) = \\ \mathbb{E}\left(\lambda \int_s^t (X_u - 1)\, du \,|\mathcal{F}_s\right) = -\lambda \int_s^t \mathbb{E}((X > u)\,|\mathcal{F}_s)\; du = -\lambda \int_s^t \mathrm{e}^{-\lambda(u-s)}(X > s)\, du = \\ \left(-\lambda \int_s^t \mathrm{e}^{-\lambda(u-s)}\; du\right)(X > s) = \left.\mathrm{e}^{-\lambda(u-s)}\right|_s^t (X > s) = \mathbb{E}(X_t - X_s \,|\mathcal{F}_s),$$

so that $\mathbb{E}(X_t - A_t \,|\mathcal{F}_s) = X_s - A_s$.

E7

Compute $\mathbb{E}\left(\phi(X_t)\,|\mathcal{F}_s\right)$ to show that the process $X.$ is a Markov process. Compute the transitions.

As $\phi(X_t) = \phi(0)(X > t) + \phi(1)(X \leq t) = \phi(1) + (\phi(0) - \phi(1)(X > t)$, the conditional expectation is

$$\mathbb{E}\left(\phi(X_t)\,|\mathcal{F}_s\right) = \phi(1)+(\phi(0)-\phi(1))\,\mathbb{E}\left((X > t)\,|\mathcal{F}_s\right) = \phi(1)+(\phi(0)-\phi(1))\mathrm{e}^{-\lambda(t-s)}(X > s) = P_{s,t}\phi(X_s),$$

with

$$P_{s,t}\phi(x) = \begin{cases} \phi(0)\mathrm{e}^{-\lambda(t-s)} + \phi(1)\left(1 - \mathrm{e}^{-\lambda(t-s)}\right) & \text{if } x = 0 \\ \phi(1) & \text{if } x = 1. \end{cases}$$

The transitions are computed as

$$\mathbb{P}\left(X_t = y|X_s = x\right) = \frac{\mathbb{P}\left(X_s = x, X_t = y\right)}{\mathbb{P}\left(X_s = x\right)} = \frac{\mathbb{E}\left((X_s = x)(X_t = y)\right)}{\mathbb{P}\left(X_s = x\right)} = \frac{\mathbb{E}\left((X_s = x)\,\mathbb{E}\left((X_t = y)\,|\mathcal{F}_s\right)\right)}{\mathbb{P}\left(X_s = x\right)}$$

and the previous formula for $\phi = \mathbf{1}_{\mathbf{y}}$.

E8

If $X_1, X_2, \ldots$ is an infinite sequence of independent random variables with common distribution $\mathrm{Exp}(\lambda)$ and $T_0 = 0,\ T_1 = X_1,\ T_2 = X_1 + X_2$, $T_3 = X_1 + X_2 + X_3\ \ldots$ i.e., $T_0 = 0$ and $T_n = T_{n-1} + X_n$, $n \geq 1$, then $T_n \uparrow +\infty$ for $n \to \infty$ a.s.

For all $a > 0$

$$\mathbb{P}\left(T_n \leq a\right) = \lambda \int_0^a \frac{(\lambda t)^{n-1}}{(n-1)!}\, \mathrm{e}^{-\lambda t}\, dt \to 0, \quad n \to \infty,$$

as $\left((\lambda t)^{n-1}/(n-1)!\right)_n$ is decreasing if $n \geq \lambda a$. Cf. [Williams 12.5].

11. Poisson process: construction

Definition

Let $X_1, X_2, \ldots$ be iid $\text{Exp}(\lambda)$ on the probability space $(S, \mathcal{S}, \mathbb{P})$ and $T_0 = 0$ and $T_n = T_{n-1} + X_n$, $n \geq 1$. Let $(D, \mathcal{D})$ be the space of trajectories on the set of times $[0, +\infty[$ which are 0 at $t = 0$ and take the value $n = 0, 1, 2 \ldots$ on the interval $[t_n, t_{n+1}[$ where $t_0 < t_1 < t_2 \ldots$ and $t_n \to \infty$ as $n \to \infty$. The *Poisson process with jump times* $(T_n)_n$ is defined at each $\omega \in \{\lim_n T_n = +\infty\}$ as the trajectory in $(D, \mathcal{D})$ with jumps $T_1(\omega), T_2(\omega), \ldots$, that is

$$N_t(\omega) = \sum_n n \left(T_n(\omega) \leq n < T_{n+1}(\omega)\right) =$$

$$\# \left\{ T_k(\omega) \middle| T_k \leq t \right\} = \sum_{n=0}^{\infty} (T_n \leq t).$$

12. Poisson process: properties from Def (0)

1. Each N_t is Poisson distributed with mean λt: $\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}$.
2. Each increment $N_t - N_s$, $s < t$, is Poisson distributed with mean $\lambda(t - s)$: $\mathbb{P}(N_t - N_s = n) = \frac{(\lambda(t-s))^n}{n!} \mathrm{e}^{-\lambda(t-s)}$.
3. Disjoint increments are independent: for all natural n and reals $t_1 < T_2 < \cdots < t_n$, the random variables $N_{t_j} - N_{t_{j-1}}$, $j = 1, \ldots, n$ and independent, hence
4. the sequence $N_{t_1}, N_{t2}, \ldots, N_{t_n}$ has the Markov property

$$\mathbb{E}\left(\phi(N_{t_n}) \,\middle|\, N_{t_{n-1}}, N_{t_{n-2}}, \ldots\right) = \sum_{n=0}^{\infty} \phi(n + N_{t_{n-1}}) \frac{(\lambda(t_n - t_{n-1})^n}{n!} \mathrm{e}^{-\lambda(T_n - t_{n-1})}$$

1. $\mathbb{P}(N_t = n) = \mathbb{E}((T_n \geq t)(t - T_n < X_{n+1})) = \mathbb{E}((T_n \leq t)\exp(-\lambda(t - T_n))) =$
$\int_0^t \mathrm{e}^{-\lambda(t-x)} \frac{\lambda^n x^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda x} dx = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!}$.

2.a $\mathbb{P}(N_s = n, N_t = n) = \mathbb{E}((T_n \geq s)(t - T_n < X_{n+1})) = \mathbb{E}((T_n \leq s)\exp(-\lambda(t - T_n))) =$
$\int_0^s \mathrm{e}^{-\lambda(t-x)} \frac{\lambda^n x^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda x} dx == \mathrm{e}^{-\lambda t} \frac{(\lambda s)^n}{n!} = \text{Poisson}(n, \lambda)$.
$\mathbb{P}(N_t - N_s = 0) = \sum_{n=0}^{\infty} \mathbb{P}(N_s = n, N_t = n) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda t} \frac{(\lambda s)^n}{n!} = \mathrm{e}^{-\lambda(t-s)} = \text{Poisson}(0; \lambda(t-s))$.

2.b $\mathbb{P}(N_s = n-1, N_t = n) = \mathbb{E}\left((T_{n-1} \leq s < T_n \leq t)(t - T_n < X_{n+1})\right) =$
$\mathbb{E}\left((T_{n-1} \leq s < T_n \leq t)\exp(-\lambda(t - T_n))\right) =$
$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_{n-1} \leq s)\mathrm{e}^{\lambda T_{n-1}}(s - T_{n-1} \leq X_n < t - T_{n-1})\mathrm{e}^{\lambda X_n}\right) =$
$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_{n-1} \leq s)\mathrm{e}^{\lambda T_{n-1}} \int_{s-T_{n-1}}^{t-T_{n-1}} \mathrm{e}^{\lambda x} \lambda \mathrm{e}^{-\lambda x} dx\right) = \lambda(t-s)\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_{n-1} \leq s)\mathrm{e}^{\lambda T_{n-1}}\right) =$
$\lambda(t-s)\mathrm{e}^{-\lambda t} \int_0^s \mathrm{e}^{\lambda x} \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} \mathrm{e}^{-\lambda x}\, dx = \lambda(t-s)\frac{(\lambda s)^{n-1}}{(n-1)!}\mathrm{e}^{-\lambda t}$.
$\mathbb{P}(N_t - N_s = 1) = \sum_{n=1}^{\infty} \mathbb{P}(N_s = n-1, N_t = n) = \sum_{n=1}^{\infty} \lambda(t-s)\frac{(\lambda s)^{n-1}}{(n-1)!}\mathrm{e}^{-\lambda t} = \lambda(t-s)\mathrm{e}^{-\lambda(t-s)} =$
$\text{Poisson}(1; \lambda(t-s))$.

2.c $\mathbb{P}(N_s = n-k, N_t = n) = \mathbb{E}\left((T_{n-k} \leq s < T_{n-k+1})(T_n \leq t)(t - T_n < X_{n+1})\right) =$
$\mathbb{E}\left((T_{n-k} \leq s < T_{n-k+1})(T_n \leq t)\exp(-\lambda(t - T_n))\right) =$
$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_{n-k} \leq s < T_{n-k+1})(T_n \leq t)\mathrm{e}^{\lambda T_n}\right) =$
$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_{n-k} \leq s < T_{n-k+1})\mathrm{e}^{\lambda T_{n-k+1}}(X_{n-k+2} + \cdots + X_{n-1} \leq t - T_{n-k+1})\mathrm{e}^{\lambda(X_{n-k+2}+\cdots+X_{n-1})}\right) =$
$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_{n-k} \leq s)(t < T_{n-k+1})\mathrm{e}^{\lambda T_{n-k+1}} \int_0^{t-T_{n-k+1}} \mathrm{e}^{\lambda y} \frac{\lambda^{k-1} y^{k-2}}{(k-2)!} \mathrm{e}^{-\lambda y}\, dy\right) =$
$\mathrm{e}^{-\lambda t} \frac{\lambda^{k-1}}{(k-1)!}\,\mathbb{E}\left((T_{n-k} \leq s)(t < T_{n-k+1})\mathrm{e}^{\lambda T_{n-k+1}}(t - T_{n-k+1})^{k-1}\right)$

14. Poisson process: filtration

Let N be a Poisson process with jump times $T_1, T_2, \ldots$. We say that N is the *counting process* of the sequence $T_1, T_2, \ldots$ We have $\{N_t \geq n\} = \{T_n \leq t\}$.

σ-algebras generated by a Poisson process

1. The σ-algebra $\sigma(N_t)$ has the generating π-system $\{\{T_n \leq t\} | n = 1, 2, \ldots\}$.
2. The σ-algebra $\mathcal{F}_t = \sigma(N_s \colon s \leq t)$ has generating π-system $\mathcal{C}_t =$

$$\{\{T_{n_1} \leq s_1, \ldots, T_{s_m} \leq s_m\} | m \in \mathbb{N}, n_1 \leq \cdots \leq n_m, s_1 < \cdots < s_m \leq t\}$$

3. A random variable Y is $\mathcal{F}_t$ measurable if, and only if,

$$Y = \sum_{k=0}^{\infty} G_k (N_t = k)$$

where G_k is a $\sigma(T_1, \ldots, T_k)$-measurable random variable, $k \in \mathbb{Z}_+$.

15. Proofs for slide 14 items 1,2

1. The sets $[a, +\infty[$, $a \in \mathbb{R}$ form a π-system for $(\mathbb{R}, \mathcal{B})$, hence $\{\{N_t \geq a\} | a \in \mathbb{R}\}$ is a π-system of $\sigma(N_t)$. As N_t actually takes values in $\mathbb{Z}_+$, the system contains set of the form $\{N_t \geq n\} = \{T_n \leq t\}$ for $n = 0, 1, \ldots$

2. The π-system for $\mathcal{F}_t$ contains sets which are finite intersections of elements of the form $\{T_n \leq s\}$ for all n and $s \leq t$, that is, for example,

$$\left\{T_{n_1} \leq t_1, T_{n_2} \leq t_2, \ldots, T_{n_m} \leq t_m\right\}$$

for all $m, n_1, \ldots, n_m \in \mathbb{N}$ and $t_1, \ldots, t_m \in \mathbb{R}_+$. If we order the t_i's in increasing order, consider the π-system property for each time, and the increasing order of the T_n's we obtain the stated subclass.

16. Proof for slide 14 item 3

We use the monotone class theorem [Williams 3.14]. Consider the set of random variables

$$\mathcal{H} = \left\{ \sum_{k=0}^{\infty} G_k (N_t = k) \middle| G_k \in \mathcal{L}^{\infty}(\mathcal{G}_k), k \in \mathbb{Z}_+ \right\},$$

with $\mathcal{G}_k = \sigma(T_1, \ldots, T_k)$. It is a vector space, it contains the constants, and it is closed for increasing limits. In fact, on each element of the partition $\{N_t = k\}$, $k \in \mathbb{Z}_+$ is a generic class of bounded random variables.
Moreover, it is an *ring*, as

$$\left(\sum_{k=0}^{\infty} G_k (N_t = k) \right) \left(\sum_{k=0}^{\infty} G'_k (N_t = k) \right) = \sum_{k=0}^{\infty} G_k G'_k (N_t = k).$$

Each indicator function of the form $(T_n \leq s)$, $n \in \mathbb{Z}_+$, $s \leq t$, belongs to $\mathcal{H}$. In fact,

$$(T_n \leq s)(N_t = k) = (T_n \leq s, T_k \leq t, T_{k+1} > t) = \begin{cases} (T_n \leq s)(N_t = k) & \text{if } n \leq k, \\ 0 & \text{if } n > k, \end{cases}$$

hence

$$(T_n \leq s) = \sum_{k \geq n} (T_n \leq s)(N_t = k) = \sum_k G_k (N_t = k),$$

with $G_k = 0$ for $k < n$ and $G_k = (T_n \leq s)$ for $k \geq n$. As $\mathcal{H}$ is closed for product, each element of the π-system $\mathcal{C}_t$ is included, and the result follows for bounded random variables. The general case is obtained by point-wise limits of bounded random variables.

17. Poisson process: conditioning

Conditioning to $\mathcal{F}_t$

Let N be a Poisson process on $(S, \mathcal{S})$ with jumps $T_1, T_2, \ldots$, and let a time $s \geq 0$ be given.

1. For each integrable random variable Y there exists a sequence $Y_k = g_k(T_1, \ldots, T_k)$, $k = 0, 1, \ldots$, such that

$$\mathbb{E}\left(Y \,|\mathcal{F}_s\right) = \sum_{k=0}^{\infty} Y_k(N_s = k).$$

2. Each $Y_n = g_n(T_1, \ldots, T_n)$, $n = 0, 1, \ldots$ is characterized by

$$\mathbb{E}\left(Y(N_s = n)G\right) = \mathrm{e}^{-\lambda s}\,\mathbb{E}\left(Y_n(T_n \leq s)\mathrm{e}^{\lambda T_n} G\right), \quad G \in \mathcal{L}^{\infty}(T_1, \ldots, T_n).$$

18. Exercise

E9

Prove the previous theorem

See [Williams 9.2]. The conditional expectation $\mathbb{E}\left(Y \mid \mathcal{F}_s\right)$ is a random variable measurable for $\mathcal{F}_s$, then $\mathbb{E}\left(Y \mid \mathcal{F}_s\right) = \sum_{k=0}^{\infty} Y_k(N_s = k)$. Note that $G_n(N_s = n)$, $G_n = g_(Y_1, \dots, Y_n)$, G_n indicator, $n = 0, 1, \dots,$ is a π-system for $\mathcal{F}_s$. Then we want

$$\mathbb{E}\left(YG_n(N_s = n)\right) = \mathbb{E}\left(\mathbb{E}\left(Y \mid \mathcal{F}_s\right) G_n(N_s = n)\right) = \mathbb{E}\left(\left(\sum_{k=0}^{\infty} Y_k(N_s = k)\right) G_n(N_s = n)\right) = \mathbb{E}\left(Y_n G_n(N_s = n)\right).$$

As

$$(N_s = n) = (T_n \leq s < T_{n+1}) = (T_n \leq s < T_n + X_{n+1}) = (T_n \leq s)(s - T_n < X_{n+1}),$$

from the independence of $(T_1, \dots, T_n)$ and X_{n+1},

$$\mathbb{E}\left(Y_n G_n(N_s = n)\right) = \mathbb{E}\left(Y_n G_n(T_n \leq s)(s - T_n < X_{n+1})\right) =$$
$$\mathbb{E}\left(Y_n G_n(T_n \leq s)\mathrm{e}^{-\lambda(s - T_n)}\right) = \mathrm{e}^{-\lambda s}\,\mathbb{E}\left(Y_n G_n(T_n \leq s)\mathrm{e}^{\lambda T_n}\right).$$

19. Markov property, independent increments

Key result

1. The Poisson process is a *Markov process* i.e., for each bounded $\phi\colon \mathbb{R} \to \mathbb{R}$ and times $s < t$ we have

$$\mathbb{E}\left(\phi(N_t)\,|\mathcal{F}_s\right) = \hat{\phi}(N_s),$$

   with transition

$$\hat{\phi}(n) = \sum_{k=n}^{\infty} \phi(k) \frac{(\lambda(t-s))^{k-n}}{(k-n)!} = \sum_{m=0}^{\infty} \phi(n+m) \frac{(\lambda(t-s))^{m}}{m!}.$$

2. It follows that the Poisson process has *independent homogeneous increments* i.e., for each bounded $\phi\colon \mathbb{R} \to \mathbb{R}$ and times $s < t$ we have

$$\mathbb{E}\left(\phi(N_t - N_s)\,|\mathcal{F}_s\right) = \mathbb{E}\left(\phi(N_{t-s})\right).$$

Ex.10

Proof of 1.

We apply the formula for conditioning to the r.v. $Y = \phi(N_t)$, that is $\mathbb{E}(\phi(N_t)|\mathcal{F}_s) = \sum_{k=0}^{\infty} Y_k(N_s = k)$, with the characterization $\mathbb{E}(\phi(N_t)(N_s = n)G_n) = \mathrm{e}^{-\lambda s}\,\mathbb{E}\left(Y_n(T_n \le s)\mathrm{e}^{\lambda T_n} G_n\right)$, $G_n = g_n(T_1, \dots, T_n)$. The RES is

$$\mathbb{E}(\phi(N_t)(N_s = n)G_n) = \sum_{k=0}^{\infty} \phi(k)\,\mathbb{E}((N_t = k)(N_s = n)G_n) = \sum_{k=n}^{\infty} \phi(k)\,\mathbb{E}((N_t = k)(N_s = n)G_n)$$

because $N_s \le N_t$. We compute each term of the RES. For $k = n$ we have

$$\mathbb{E}((N_t = n)(N_s = n)G_n) = \mathbb{E}((T_n \le s)(t < T_{n+1})G_n) =$$
$$\mathbb{E}((T_n \le s)(t - T_n < X_{n+1})G_n) = \mathbb{E}\left((T_n \le s)\mathrm{e}^{-\lambda(t-T_n)G_n}\right) = \mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{-\lambda T_n} G_n\right).$$

If $k = n + 1$,

$$\mathbb{E}((N_t = n+1)(N_s = n)G_n) = \mathbb{E}((N_s = n)(T_{n+1} \le t < T_{n+2})G_n) =$$
$$\mathbb{E}((N_s = n)(T_{n+1} \le t < T_{n+1} + X_{n+2})G_n) = \mathbb{E}((N_s = n)(T_{n+1} \le t)(t - T_{n+1} < X_{n+2})G_n) =$$
$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((N_s = n)(T_{n+1} \le t)\mathrm{e}^{\lambda T_{n+1}} G_n\right) = \mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s < T_n + X_{n+1})(T_n + X_{n+1} \le t)\mathrm{e}^{\lambda(T_n + X_{n+1})} G_n\right) =$$
$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{\lambda T_n}(s - T_n < X_{n+1})(T_n \le t)(X_{n+1} \le t - T_n)\mathrm{e}^{\lambda X_{n+1}} G_n\right) =$$
$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{\lambda T_n}(s - T_n < X_{n+1} \le t - T_n)\mathrm{e}^{\lambda X_{n+1}} G_n\right) = \lambda(t - s)\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{\lambda T_n} G_n\right)$$

If $k = n + h$, $h > 1$,

$$\mathbb{E}\left((N_t = n+h)(N_s = n)G_n\right) = \mathbb{E}\left((N_s = n)(T_{n+h} \le t < T_{n+h+1})G_n\right) =$$

$$\mathbb{E}\left((N_s = n)(T_{n+h} \le t < T_{n+h} + X_{n+h+1})G_n\right) = \mathbb{E}\left((N_s = n)(T_{n+h} \le t)(t - T_{n+h} < X_{n+h+1})G_n\right) =$$

$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((N_s = n)(T_{n+h} \le t)\mathrm{e}^{\lambda T_{n+k}}G_n\right) =$$

$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((N_s = n)(T_{n+1} + (X_{n+2} + \cdots + X_{n+h}) \le t)\mathrm{e}^{\lambda(T_{n+1}+(X_{n+2}+\cdots+X_{n+h}))}G_n\right) =$$

$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((N_s = n)\mathrm{e}^{\lambda T_{n+1}}(T_{n+1} \le t)(X_{n+2} + \cdots + X_{n+h} \le t - T_{n+1})\mathrm{e}^{\lambda(X_{n+2}+\cdots+X_{n+h})}G_n\right) =$$

$$\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((N_s = n)\mathrm{e}^{\lambda T_{n+1}}(T_{n+1} \le t)\left(\int_0^{t-T_{n+1}} \mathrm{e}^{\lambda x}\frac{\lambda^{h-1}x^{h-2}}{(h-2)!}\mathrm{e}^{-\lambda x}dx\right)G_n\right) =$$

$$\frac{\lambda^{h-1}}{(h-1)!}\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((N_s = n)\mathrm{e}^{\lambda T_{n+1}}(T_{n+1} \le t)(t - T_{n+1})^{h-1}G_n\right) =$$

$$\frac{\lambda^{h-1}}{(h-1)!}\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s < T_{n+1})\mathrm{e}^{\lambda T_{n+1}}(T_{n+1} \le t)(t - T_{n+1})^{h-1}G_n\right) =$$

$$\frac{\lambda^{h-1}}{(h-1)!}\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{\lambda T_n}(s - T_n < X_{n+1} \le t - T_n)\mathrm{e}^{\lambda X_{n+1}}(t - T_n - X_{n+1})^{h-1}G_n\right) =$$

$$\frac{\lambda^{h-1}}{(h-1)!}\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{\lambda T_n}\left(\int_{s-T_n}^{t-T_n} \mathrm{e}^{\lambda x}(t - T_n - x)^{h-1}\lambda\mathrm{e}^{-\lambda x}dx\right)G_n\right) =$$

$$\frac{(\lambda(t-s))^h}{h!}\mathrm{e}^{-\lambda t}\,\mathbb{E}\left((T_n \le s)\mathrm{e}^{\lambda T_n}G_n\right)$$

Collecting all cases

$$\mathbb{E}\left(\phi(N_t)(N_s = n)G_n\right) = \mathrm{e}^{-\lambda t}\,\mathbb{E}\left(\left(\sum_{k=n}^{\infty}\phi(k)\frac{(\lambda(t-s))^{k-n}}{(k-n)!}\right)(T_n \leq s)\mathrm{e}^{\lambda T_n}G_n\right).$$

It follows $Y_n = \sum_{k=n}^{\infty}\phi(k)\frac{(\lambda(t-s))^{k-n}}{(k-n)!}$ and

$$\mathbb{E}\left(\phi(N_t)\,|\mathcal{F}_s\,\right) = \sum_{n=0}^{\infty}(N_s = n)\sum_{k=n}^{\infty}\phi(k)\frac{(\lambda(t-s))^{k-n}}{(k-n)!} = \sum_{n=0}^{\infty}\hat{\phi}(n)(N_s = n) = \hat{\phi}(N_s)$$

Ex.11

Proof of 2.

E12Telegrapher process.

Define $Y_t = \int_0^t (-1)^{N_u}\, du$. Is it independent increments? A sub-martingale? A Markov process?

Poisson process as a Lévy process

Equivalent definition (1)

A counting process $N(t)$, $t \geq 0$ is a *Poisson process* with intensity $\lambda > 0$ if, and only if,

1. $N(0) = 0$
2. The process has independent increments.
3. The number of events occurring in the time interval $]s, t]$, with length $t - s$, has Poisson distribution with mean $\lambda(t - s)$:

$$\mathbb{P}\left(N(t) - N(s) = n\right) = \mathrm{e}^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!} \quad n = 0, 1, 2$$

R In particular, such a process has stationary increments and is continuous in probability.

Ex.13

Proof of Poisson (1) is equivalent to Poisson (0).

The "only if" part is already proved. Let us compute the joint finite distributions of the process N. For $t_1 < \cdots < t_n$ the random variable $(N_{t_1}, \ldots, N_{t_n})$ has values in $\left\{\mathbf{k} \in \mathbb{Z}_+^n \middle| k_1 \leq \cdots \leq k_n\right\}$ and (discrete) density

$$
\begin{aligned}
\mathbb{P}\left(N_{t_1} = k_1, \ldots, N_{t_n} = k_n\right) &= \mathbb{P}\left(N_{t_1} = k_1, \ldots, N_{t_n} - N_{t_{n-1}} = k_n - k_{n-1}\right) \\
&= \prod_{j=1}^{n} \mathbb{P}\left(N_{t_j} - N_{t_{j-1}} = k_j - k_{j-1}\right) \qquad k_0 = 0,\, t_0 = 0 \\
&= \prod_{j=1}^{n} \mathrm{e}^{-\lambda(t_j - t_{j-1})} \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} \\
&= \mathrm{e}^{-\lambda t_n} \prod_{j=1}^{n} \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!}.
\end{aligned}
$$

With $t_j = s_1 + \cdots + s_j$, $k_j = h_1 + \cdots + h_j$, we have $s_j > 0$ and $h_j \geq 0$, for $j = 1, \ldots, n$, hence

$$
\mathbb{P}\left(N_{s_1} = h_1, \ldots, N_{s_1 + \cdots + s_n} = h_1 + \cdots + h_n\right) = \prod_{j=1}^{n} \mathrm{e}^{-\lambda s_j} \frac{(\lambda s_j)^{h_j}}{h_j!} = \mathrm{e}^{-\lambda(s_1 + \cdots + s_n)} \prod_{j=1}^{n} \frac{(\lambda s_j)^{h_j}}{h_j!}.
$$

Let us compute the joint distribution of the arrival times. The vector $(T_1, \ldots, T_n)$ takes values in $\{\mathbf{t} | 0 < t_1 < \cdots < t_n\}$, which in turn is the image of $\mathbb{R}_>^n$ under the cumulative sum map sum: $\mathbf{x} \mapsto (x_1, \dots, x_1 + \cdots + x_n)$. As $\left\{]s_1, +\infty[\times \cdots \times]s_n, +\infty[\middle| s_j > 0\right\}$ is a π-system, its cumulative sum image is a π-system $\left\{]t_1, +\infty[\times \cdots \times]t_n, +\infty[| 0 < t_1 < \cdots < t_n\right\}$. Let us compute the probability of each element

$$\begin{aligned}
\mathbb{P}(T_1 > t_1, \dots, T_n > t_n) &= \mathbb{P}\left(N_{t_1} < 1, \dots, N_{t_n} < n\right) \\
&= \sum_{k_2 \leq \cdots \leq k_n, k_j < j} \mathbb{P}\left(N_{t_1} = 0, \dots, N_{t_n} = k_n\right) \\
&= \sum_{k_2 \leq \cdots \leq k_n, k_j < j} \mathrm{e}^{-\lambda t_n} \prod_{j=2}^{n} \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!}
\end{aligned}$$

This equation uniquely identifies the joint distribution of $T_1, \dots, T_n$, hence the joint distribution of the inter-times $X_j = T_j - T_{j-1}$, $j = 1, \dots, n$. Assume first $n = 1$,

$$\mathbb{P}(T_1 > t_1) = \mathbb{P}\left(N_{t_1} < 0\right) = \mathrm{e}^{-\lambda t_1}, \quad -\frac{\partial}{\partial t_1} \mathbb{P}(T_1 > t_1) = \lambda \mathrm{e}^{-\lambda t_1}.$$

If $n = 2$,

$$\begin{aligned}
\mathbb{P}(T_1 > t_1, T_2 > t_2) = \mathbb{P}\left(N_{t_1} < 1, N_{t_2} < 2\right) = \mathbb{P}\left(N_{t_2} = 0\right) + \mathbb{P}\left(N_{t_1} = 0, N_{t_2} = 1\right) = \\
\mathrm{e}^{-\lambda t_2} + \mathrm{e}^{-\lambda t_1} \mathrm{e}^{-\lambda(t_2 - t_1)} \lambda(t_2 - t_1) = \mathrm{e}^{-\lambda t_2}(1 + \lambda(t_2 - t_1)),
\end{aligned}$$

and

$$-\frac{\partial}{\partial t_1} \mathbb{P}(T_1 > t_1, T_2 > t_2) = \lambda \mathrm{e}^{-\lambda t_2}, \quad (-1)^2 \frac{\partial^2}{\partial t_1 \partial t_2} \mathbb{P}(T_1 > t_1, T_2 > t_2) = \lambda^2 \mathrm{e}^{-\lambda t_2}.$$

If $n = 3$ we have a summation of $\mathbb{P}\left(N_{t_1} = k_1, N_{t_2} = k_2, N_{t_3} = k_3\right)$ over the cases $k_1 \leq k_2 \leq k_3$, $k_1 < 1$, $k_2 < 2$, $k_3 < 3$, that is

k_1	0	0	0	0	0
k_2	0	0	0	1	1
k_3	0	1	2	1	2

The time t_1 appears in $\mathbb{P}\left(N_{t_1} = k_1, N_{t_2} = k_2, N_{t_3} = k_3\right)$ only if $k_2 = 1$, hence

$$-\frac{\partial}{\partial t_1}\mathbb{P}\left(T_1 > t_1, T_2 > t_2, T_3 > t_3\right) = -\mathrm{e}^{-\lambda t_3}\frac{\partial}{\partial t_1}\left(\lambda(t_2 - t_1) + \lambda(t_2 - t_1)\lambda(t_3 - t_2)\right) =$$
$$-\mathrm{e}^{-\lambda t_3}(1 + \lambda(t_3 - t_2))\frac{\partial}{\partial t_1}\lambda(t_2 - t_1) = \lambda\mathrm{e}^{-\lambda t_3}(1 + \lambda(t_3 - t_2))$$

so that the recursion is apparent and gives

$$(-1)^3\frac{\partial^3}{\partial t_1\partial t_2\partial t_3}\mathbb{P}\left(T_1 > t_1, T_2 > t_2, T_3 > t_3\right) = \lambda^3\mathrm{e}^{-\lambda t_3}.$$

Variant definition (2)

A counting process $N(t)$, $t \geq 0$ is a *Poisson process* with intensity $\lambda > 0$ if

1. $N(0) = 0$
2. The process has independent and stationary increments.
3. $\mathbb{P}(N(t) = 1) = \lambda t + o(t)$ as $t \to 0$
4. $\mathbb{P}(N(t) > 1) = o(t)$ as $t \to 0$

$\mathbb{E}(N(t)) = \lambda t$ and for $t \to 0$:

$$\begin{cases} \mathbb{P}(N(t) = 1) = \mathrm{e}^{-\lambda t}(\lambda t) = \lambda t + o(t) \\ \mathbb{P}(N(t) > 1) = 1 - \mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t}(\lambda t) = 1 - \mathrm{e}^{-\lambda t}(1 + (\lambda t)) = o(t) \end{cases}$$

Ex.14

The two definitions are equivalent.

Let us consider the moment generating function of $N(t)$, i.e. $g(t) = \mathbb{E}\left(e^{uN(t)}\right)$. Note $g(0) = 1$, and let us derive a differential equation for g.

$$\begin{aligned} g(t+h) &= \mathbb{E}\left(e^{uN(t+h)}\right) \\ &= \mathbb{E}\left(e^{uN(t)} e^{u(N(t+h)-N(t))}\right) \quad \text{independence} \\ &= \mathbb{E}\left(e^{uN(t)}\right) \mathbb{E}\left(e^{u(N(t+h)-N(t))}\right) \quad \text{stationarity} \\ &= g(t)g(h) \end{aligned}$$

As $h \to 0$

$$\begin{aligned} g(h) &= \sum_k e^{uk}\, \mathbb{P}(N(h) = k) \\ &= (1 - \lambda + o(h)) + e^u(\lambda h + o(h)) + \sum_{k>1} e^{uk}\, o(h) \\ &= \boxed{1 - (\lambda h) + e^u(\lambda h) + o(h)} \end{aligned}$$

By letting $h \to 0$ in the definition of derivative we obtain the equation

$$g'(t) = \lambda(e^u - 1)g(t)$$

whose solution is

$$\boxed{g(t) = \exp\left[\lambda\left(e^u - 1\right)\right]}$$

This is the moment generating function of the Poisson(λt) distribution.

Ex.15

Check the following:

- prove the last statement;
- take a subdivision of the interval $[0, t]$ and use the binomial law to count how many contain a jump of $N(t)$;
- avoid the stationarity condition by introducing a new condition infinitesimal condition.

30. Poisson process: simulation

- The *conditional distribution of a random variable X given the event B* is the probability measure $\mu_{X|B}$ of the domain of X such that for all integrable ϕ it holds $\mathbb{E}(\phi \circ X \, \mathbb{1}_B) = \mathbb{P}(B) \int \phi(x) \, \mu_{X|B}(dx)$. Note that $\mu_{X|B}$ is supported by B and the notation $\mathbb{E}(\phi \circ X \,|B\,) = \int \phi(x) \, \mu_{X|B}(dx)$. The conditional distribution is identified on a monotone class.
- Let $U_1, \ldots, U_n$ be iid $\mathrm{U}(0,t)$. The sorting map $\tau(t_1, \ldots, t_n) = (t_{(1)}, \ldots, t_{(n)})$ is almost surely defined and takes values in the open simplex $\Delta(t) = \{\mathbf{s} = (s_1, \ldots, s_n) | 0 < s_1 < \cdots < s_n < t\}$. The $\Delta(t)$-valued random variable $(U_{(1)}, \ldots, U_{(n)}) = \tau(U_1, \ldots, U_n)$ is the *order statistics* of $(U_1, \ldots, U_n)$. The distribution ν of the order statistics is the image under the sorting map of the uniform distribution, $\int_{\Delta(t)} \phi(\mathbf{s}) \, \nu(d\mathbf{s}) = \int_{\Delta(t)} \phi(\mathbf{s}) \, \tau_* \nu(d\mathbf{s}) = t^{-n} \int_{[0,t]^n} \phi \circ \tau(\mathbf{t}) \, d\mathbf{t} =$ $t^{-n} \sum_{\Pi \in \mathfrak{P}} \int_{\Pi^{-1}(\Delta(t))} \phi \circ \tau(\mathbf{t}) \, d\mathbf{t} = n! t^{-n} \int_{\Delta(t)} \phi(\mathbf{t}) \, d\mathbf{t}$, where $\mathfrak{P}$ is the group of permutation matrices on $\mathbb{R}^n$.

Past jump times are uniform and independent

Let $T_1, T_2, \ldots$ be the arrival times of a Poisson process with intensity λ.

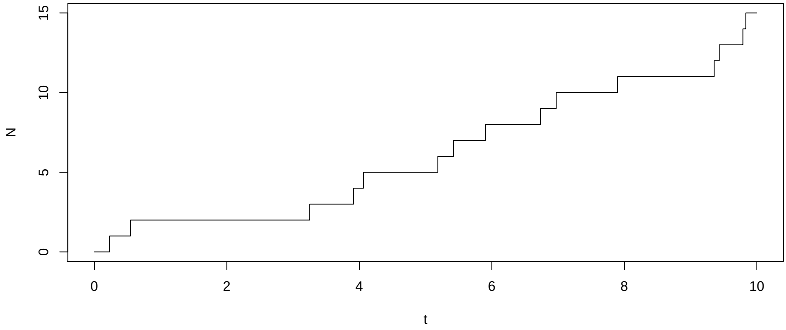
- The joint distribution of $T_1, \ldots, T_n$, conditional to $\{N(t) = n\}$ has uniform density

$$f(t_1, \ldots, t_n) = \frac{n!}{t^n}(0 < t_1 < \cdots < t_n < t)$$

- Such a density is the joint density of the order statistics of n random variables $U_1, \ldots, U_n$ IID uniformly distributed on $]0, t[$.

A simulation

```
lambda <- 1; t <- 10
n <- rpois(1,lambda*t); uniform <- runif(n,0,t)
x <- sort(uniform)
plot(c(0,x,t),c(0:n,n),type="s",xlab="t",ylab="N")
```



Ex.16

Proof.

Consider the monotone class $\{\mathbf{t} \mapsto \phi_1(t_1)\cdots\phi_n(t_n)|\phi_i \text{bounded}\}$.

$$\begin{aligned}
\mathbb{E}\left(\phi_1(T_1)\cdots\phi_n(T_n)(N_t = n)\right) &= \mathbb{E}\left(\phi_1(T_1)\cdots\phi_n(T_n)(N_t = n)\right) \\
&= \mathbb{E}\left(\phi_1(T_1)\cdots\phi_n(T_n)(T_n \le t)(t - T_n < X_{n+1})\right) \\
&= \mathrm{e}^{-\lambda t}\,\mathbb{E}\left(\phi_1(T_1)\cdots\phi_n(T_n)(T_n \le t)\mathrm{e}^{\lambda T_n}\right) \\
&= \mathrm{e}^{-\lambda t}\,\mathbb{E}\left(\phi_1(X_1)\cdots\phi_n(X_1 + \cdots + X_n)(X_1 + \cdots + X_n \le t)\mathrm{e}^{\lambda(X_1 + \cdots + X_n)}\right) \\
&= \lambda^n \mathrm{e}^{-\lambda t} \int\cdots\int_{]0,\infty[^n} \phi_1(x_1)\cdots\phi_n(x_1 + \cdots + x_n)(x_1 + \cdots + x_n \le t)\; dx_1\cdots dx_n \\
&= \\
&= \lambda^n \mathrm{e}^{-\lambda t} \int\cdots\int_{\Delta(t)} \phi_1(s_1)\cdots\phi_n(s_n)\; dt_1\cdots dt_n \\
&= \frac{(\lambda t)^n}{n!}\mathrm{e}^{-\lambda t} \int\cdots\int \phi_1(s_1)\cdots\phi_n(s_n)\frac{n!}{t^n}\mathbb{1}_{\Delta(t)}\; dt_1\cdots dt_n.
\end{aligned}$$