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4

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Martingale

Definition (Stochastic process)

A discrete-time stochastic process on the probability space $(\Omega, \mathcal{F}, \mu)$ is a family $(X_t)_{t\in I}$, $I \subset \mathbb{Z}$, of random variables.

Definition (Markov process)

A stochastic process is a Markov process if for each bounded measurable real function ϕ and all $t \in I$ it holds

$$\mathsf{E}_\mu\left(\phi \circ X_t | X_s \colon s < t\right) = \mathsf{E}_\mu\left(\phi \circ X_t | X_{t-}\right), \quad t- = \sup\left\{s \in I | s < t\right\}$$

Definition (Martingale)

A martingale is a real stochastic process $(X_t)_{t\in I}$ such that

1. $\mathsf{E}_\mu\left(|X_t|\right) < +\infty$,
2. $\mathsf{E}_\mu\left(X_t | X_s \colon s < t\right) = X_{t-}$, $t, t- \in I$.

Filtration

Definition

1. Given a probability space $(\Omega, \mathcal{F}, \mu)$ and a set of discrete times $I \subset \mathbb{Z}$, a filtration is an increasing family $(\mathcal{F}_t)_{t \in I}$ of sub-σ-algebras of $\mathcal{F}$. The tuple $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t \in I})$ is a filtered probability space.
2. The natural filtration the stochastic process $(X_t)_{t \in I}$ is the filtration $(\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma \{X_s | s \leq t\}$.
3. Given a filtered probability space $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t \in I})$, a stocastic process $(X_t)_{t \in I}$ is adapted if X_t is $\mathcal{F}_t$-measurable for all $t \in I$.

- Ch. 10 of D. Williams. *Probability with martingales*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991.

Basic facts about martingales

- If $(X_t)_{t \in I}$ is a martingale, then $t \mapsto \mathsf{E}_\mu(X_t)$ is constant.
- $(X_t)_{t \in I}$ with $\mathsf{E}_\mu(|X_t|) < \infty$, $t \in I$, is a martingale if, and only if

$$\mathsf{E}_\mu(X_t - X_{t-} | X_s \colon s \leq t-) = 0, \quad t, t- \in I \ .$$

- Let $(X_t)_{t \in I}$ be adapted to $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t \in I})$, real and integrable. Then $(X_t)_{t \in I}$ is a martingale if, and only if,

$$\mathsf{E}_\mu(X_t | \mathcal{F}_f) = X_{t-}, \quad t, t- \in I \ .$$

- A process $(A_t)_{t \in I}$ is previsible if each A_t is $\mathcal{F}_{t-}$-measurable or constant if $t-$ is not defined. In particular, a previsible process is adapted. A previsible martingale is constant.
- Let $(Y_t)_{t=0}^n$ be a real integrable stochastic process adapted to $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t=0}^n)$. There exists a previsible real integrable stochastic process $(A_t)_{t=0}^n$ such that $X_t = Y_t - A_t$ is a martingale. Such a process A is called a compensator of Y.

Examples of martingales

1. Let $X_t,\ t = 1, 2, \ldots, N$ be independent with $\mathsf{E}_\mu(X_t) = 0$. Then $S_t = \sum_{s \le t} X_t,\ t = 1, 2, \ldots, n$ is a martingale.
2. Let $X_1, X_2, \ldots, X_N$ be a Gaussian martingale. Then $(X_{t+1} - X_t)$, $t = 1, 2, \ldots, (N-1)$ are independent.
3. Let $X_t,\ t = 1, 2, \ldots, N$ be nonnegative and independent with $\mathsf{E}_\mu(X_t) = 1$. Then $Y_t = \prod_{s \le t} X_t,\ t = 1, 2, \ldots, n$ is a martingale.
4. Let $(X_t)_{t=0}^{\infty}$ be a martingale and $(C_t)_{t=1}^{\infty}$ previsible and bounded. Then $Y_t = \sum_{s=1}^{t} C_s(X_s - X_{s-1})$ is a martingale.
5. Let $(X_t)_{t=0}^{n}$ be a Markov process and write $\mathsf{E}_\mu(\phi(X_t) | X_0, \ldots, X_{t-1}) = (P_t\phi)(X_{t-1})$. Then for each ϕ bounded

$$\phi(X_t) - \sum_{s=1}^{t} (P_s - I)\phi(X_{s-1})$$

is a martingale.

Examples of martingales: proofs

1. As $S_1 = X_1$ and $S_t - S_{t-1} = X_t$, we have $\mathcal{F}_t = \sigma(S_s \colon s \leq t) = \sigma(X_s \colon s \leq t)$, hence $\mathrm{E}\left(S_t - S_{t-1} | \mathcal{F}_{t-1}\right) = \mathrm{E}\left(X_t | X_s \colon s < t\right) = \mathrm{E}\left(X_t\right) = 0$.
2. As $\mathrm{E}\left(X_t - X_{t-1}\right) = 0$, then $\mathrm{Cov}\left(X_s - X_{s-1}, X_t - X_{t-1}\right) = \mathrm{E}\left((X_s - X_{s-1})(X_t - X_{t-1})\right)$. Choose $s < t$. Then $\mathrm{Cov}\left(X_s - X_{s-1}, X_t - X_{t-1}\right) = \mathrm{E}\left((X_s - X_{s-1})\, \mathrm{E}\left(X_t - X_{t-1} | X_u \colon u \leq (t-1)\right)\right) = 0$, hence $X_s - X_{s-1} \perp\!\!\!\perp X_t - X_{t-1}$.
3. Take $\mathcal{F}_t = \sigma(X_s \colon s \leq t)$. Then $\mathrm{E}\left(Y_t | \mathcal{F}_{t-1}\right) = Y_{t-1}\, \mathrm{E}\left(X_t | X_s \colon s < t\right) = Y_{t-1}$.
4. $\mathrm{E}\left(Y_t - Y_{t-1} | \mathcal{F}_{t-1}\right) = \mathrm{E}\left(C_t(X_t - X_{t-1}) | \mathcal{F}_{t-1}\right) = C_t\, \mathrm{E}\left(X_t - X_{t-1} | \mathcal{F}_{t-1}\right) = 0$.
5. $\mathrm{E}\left(\phi(X_t) | X_s \colon s < t\right) = \mathrm{E}\left(\phi(X_t) - \phi(X_t - 1) | X_s \colon s < t\right) + \phi(X_t - 1) = \mathrm{E}\left((P_t - I)\phi(X_{t-1} | X_s \colon s < t\right) + \phi(X_t - 1) = P_t(X_{t-1})$. [And conversely!]

Stopping time

Definition

Given the filtered probability space $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t)_{t \in I})$, $I \subset \mathbb{Z}$, a random time is an $\mathcal{F}$-measurable mapping $T \colon \Omega \to I \cup \{+\infty\}$. A random time is a stopping time (or an optional time) if

$$\{T \leq t\} = \{\omega \in \Omega | T(\omega) \leq t\} \in \mathcal{F}_t, \quad t \in I \ .$$

Equivalently, $\{T = t\} \in \mathcal{F}_t$ or $\{T > t\} \in \mathcal{F}_t$.

First visit

Let $(X_t)_{t \in I}$ be adapted, $X_t \colon \Omega \to S$, and $B \subset S$ measurable. The random time

$$T(\omega) = \inf \{s \in I | X_s(\omega) \in B\} , \quad \inf \emptyset = +\infty \ ,$$

is a stopping time. In fact,

$$\{\omega \in \Omega | T(\omega) \leq t\} = \cup_{s \leq t} \{\omega \in \Omega | X_s(\omega) \in B\} \ .$$

Properties of stopping times

By recoding $I \subset \mathbb{Z}$ we can assume I to be an interval of $\mathbb{Z}$.

- A constant time $T = \bar{t}$ is a stopping time. In fact, $\{\omega \in \Omega | T \leq t\}$ is either $\emptyset$ or ω.
- Given $B \in \mathcal{F}_{\bar{t}}$ the time $T(\omega) = \bar{t}$ if $\omega \in B$ and $+\infty$ otherwise is a stopping time. In fact, $\{\omega \in \Omega | T \leq t\}$ if $\emptyset$ if $t < \bar{t}$ and B if $t \geq \bar{t}$.
- If S and T are stopping times, then $S \wedge T$ and $S \vee T$ are stopping times. In fact,
  $\{\omega \in \Omega | S \wedge T \leq t\} = \{\omega \in \Omega | S \leq t\} \cup \{\omega \in \Omega | T \leq t\}$ and
  $\{\omega \in \Omega | S \vee T \leq t\} = \{\omega \in \Omega | S \leq t\} \cap \{\omega \in \Omega | T \leq t\}$
- If $I \subset \mathbb{Z}_{\geq}$ and both S and T are stopping times, then $S + T$ (defined to take value $S(\omega) + T(\omega)$ if in I, $+\infty$ otherwise) is a stopping time. In fact, for each $u \in I$,

  $$\{S + T = u\} = \cup_{s,t \in I, s+t=u} (\{S = s\} \cap \{T = t\}) \in \mathcal{F}_u \ ,$$

  because $s, t \geq 0$ and $s + t = u$ implies $s, t \leq u$.

Stopped process

Definition

Let $(X_t)_{t \in I}$ be an adapted process and T a finite stopping time.

- $X_T = (\omega \mapsto X_{T(\omega)}(\omega))$ is a random variable, which is integrable if T is bounded;
- the stopped process X^T is the adapted process defined by $(X^T)_t(\omega) = X_{T(\omega) \wedge t}(\omega)$, and it is integrable if T is bounded below.

- It is better to think to the stochastic process as a function $X \colon \Omega \times I$, $(\omega, t) \mapsto X(\omega, t) = X_t(\omega)$. Then X_T is the composed function $\omega \mapsto (\omega, T(\omega)) \mapsto X(\omega, T(\omega))$ and the stopped process is defined by $X^T = X$ on the set $\{(\omega, t) | T(\omega) \geq t\}$ and equal to X_T otherwise.
- For any stopping time T, the real process $C_t = \mathbf{1}_{\{T \leq t\}}$ is adapted.
- For any stopping time T, the real process $C_t = \mathbf{1}_{\{T \geq t\}}$ is previsible.

Martingales and stopping times I

Theorem (Doob)

1. *A process* $(X_t)_{t\in I}$ *is a martingale if, and only if,* $\mathrm{E}(X_T)$ *is constant for each bounded stopping time* T.
2. *If* $(X_t)_{t\in I}$ *is a martingale, and* T *is a stopping time bounded below, then the stopped process is a martingale.*

Proof of 1.

Let X be a martingale and T a stopping time with $t_0 \le T \le t_1$. Then

$$\mathrm{E}\left(\sum_{t=t_0}^{t_1} X_t \mathbf{1}_{\{T=t\}}\right) = \sum_{t=t_0}^{t_1} \mathrm{E}\left(X_t \mathbf{1}_{\{T=t\}}\right) = \sum_{t=t_0}^{t_1} \mathrm{E}\left(X_{t_1} \mathbf{1}_{\{T=t\}}\right) = \mathrm{E}\left(X_{t_1}\right) \ .$$

Conversely, for each $t, t-1 \in I$ and $B \in \mathcal{F}_{t-1}$ consider the stopping times $S = (t-1)\mathbf{1}_B + t\mathbf{1}_{B^c}$ and $T = t$.

$$0 = \mathrm{E}(X_T) - \mathrm{E}(X_S) = \mathrm{E}(X_T - X_S) = \mathrm{E}(\mathbf{1}_B(X_t - X_{t-1})) \ .$$

□

Martingales and stopping times II

Proof of 2.

We check that for each bounded stopping time S,

$$\mathsf{E}\left((X^T)_S\right) = \mathsf{E}\left(X_{S\wedge T}\right)$$

is constant. Other proof: If $t_0 \leq T$, then

$$\begin{aligned} X^T(t) &= \sum_{s=t_0+1}^{t} \left((X^T)_s - (X^T)_{s-1}\right) \\ &= \sum_{s=t_0+1}^{t} \mathbf{1}_{T\geq s}(X_s - X_{s-1}) \end{aligned}$$

and the process $C_t = \mathbf{1}_{\{T\geq t\}}$ is previsible and bounded. □

Exercise: If $(X_t)_{t=0}^{\infty}$ is Markov and T is a stopping time, then X^T is Markov.

Sub-martingale

Definition

An adapted real integrable stochastic process $(X_i)_{t \in I}$ is a sub-martingale if $s \leq t$ implies $\mathsf{E}(X_t | \mathcal{F}_s) \geq X_s$. Equivalently, a previsible compensator of X is increasing.

- Assume X is a L^2 martingale and consider the integrable process $Y_t = (X_t)^2$. Then for $s \leq t$ Jensen implies

$$\mathsf{E}(Y_t | \mathcal{F}_s) = \mathsf{E}\left((X_t)^2 \middle| \mathcal{F}_s\right) \geq \left(\mathsf{E}(X_t | \mathcal{F}_s)\right)^2 = (X_s)^2 = Y_s$$

- If X is a sub-martingale with previsible compensator A and S, T are bounded stopping times with $S \leq T$, then

$$\mathsf{E}(X_T - X_S) = \mathsf{E}(A_T - A_S) \geq 0 \ .$$