STOCHASTIC PROCESSES 2015

3. EXAMPLES OF GAUSSIAN PROCESSES

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Contents

Books

NP: Textbook [2]
GM: Monograph by S. Lauritzen on *Graphical Models* [1]

1. AR(1)

Consider a Gaussian space $(\Omega, \mathcal{F}, \mathrm{P}, (Z_n)_{n\in\mathbb{Z}_+})$. A process $(X_t)_{t\in\mathbb{Z}_+}$ is *adapted* if each X_t is measurable w.r.t the σ-algebra generated by $Z_0, \dots, Z_{t-1}, Z_t$. An adapted process is a *Gaussian* AR(1) process if it satisfies the recursive equation

$$\begin{aligned} X_0 &= \mu_0 + \sigma Z_0 \ , \\ X_t &= c + \phi X_{t-1} + \sigma_\epsilon Z_t, \quad t = 1, 2, \dots \ , \end{aligned}$$

with $1 > \sigma > 0$ and $c \in \mathbb{R}$. Each X_t is a Gaussian random variable in $\mathrm{Span}\,(Z_0, Z_1, \dots, Z_n)$. The expected value $\mu_t = \mathrm{E}\,(X_t)$, $t \geqslant 0$ is defined recursively by

$$\begin{aligned} \mu_0 &= \mu_0 \ , \\ \mu_t &= c + \phi\mu_{t-1} \ . \end{aligned}$$

There is a stationary value μ for the mean, namely $\mu = c + \phi\mu$ if $\mu = c/(1-\phi)$. It follows for all $t \geqslant 1$ that

$$(\mu_t - \mu) = \phi(\mu_{t-1} - \mu) = \cdots = \phi^t(\mu_0 - \mu) \to 0$$

as $t \to \infty$. The recursive equations for the centered random variables $X_t - \mu_t$, $t \geqslant 0$, are

$$\begin{aligned} (X_0 - \mu_0) &= \sigma Z_0 \ , \\ (X_t - \mu_t) &= \phi(X_{t-1} - \mu_{t-1}) + \sigma_\epsilon Z_t, \quad t = 1, 2, \dots \end{aligned}$$

Date: Moncalieri, May 22, 2015.

Note that the recursive equation are a description of the joint Gaussian distribution through an affine transformation to white noise:

$$\begin{aligned}
Z_0 &= \sigma^{-1}(X_0 - \mu_0) \\
Z_1 &= \sigma_\epsilon^{-1}\left((X_1 - \mu_1) - \phi(X_0 - \mu_0)\right) \\
&\vdots \\
Z_t &= \sigma_\epsilon^{-1}\left((X_t - \mu_t) - \phi(X_{t-1} - \mu_{t-1})\right)
\end{aligned}$$

or, in matrix form,

$$\begin{bmatrix} Z_0 \\ Z_1 \\ \dots \\ Z_t \end{bmatrix} = \begin{bmatrix} \sigma^{-1} & 0 & 0 & \cdots \\ -\phi\sigma_\epsilon^{-1} & \sigma_\epsilon^{-1} & 0 & \cdots \\ & \vdots & & \\ \cdots & 0 & -\phi\sigma_\epsilon^{-1} & \sigma_\epsilon^{-1} \end{bmatrix} \begin{bmatrix} X_0 - \mu_0 \\ X_1 - \mu_1 \\ \vdots \\ X_t - \mu_t \end{bmatrix}$$

The variance $\sigma_t^2 = \operatorname{Var}(X_t)$ satisfies the recursive equation

$$\begin{aligned}
\sigma_0^2 &= \operatorname{Var}(X_o) \\
\sigma_t^2 &= \phi^2\sigma_{t-1}^2 + \sigma_\epsilon^2
\end{aligned}$$

The invariant variance is characterised by $\sigma^2 = \phi^2\sigma^2 + \sigma_\epsilon^2$, that is $\sigma^2 = \sigma_\epsilon^2/(1-\phi^2)$. Note the convergence $\sigma_t^2 \to \sigma^2$.

From now on, *we assume that the mean and the variance are stationary*, as it is always true in the long run, that is, given $\sigma_\epsilon > 0$, $0 < \phi < 1$, $c \in \mathbb{R}$,

$$\begin{aligned}
X_0 &= \frac{c}{1-\phi} + \sqrt{\frac{\sigma_\epsilon^2}{1-\phi^2}} Z_0 \ , \\
X_t &= c + \phi X_{t-1} + \sigma_\epsilon Z_t, \quad t = 1, 2, \dots
\end{aligned}$$

Stationarity of the mean and the variance will be shown to imply the stationarity of all joint distributions.

1.1. **Density.** As the affine mapping $x_0, x_1, \dots, x_n \leftrightarrow (z_0, z_1, \dots, z_n)$ is 1-to-1,

$$\begin{bmatrix} \frac{\sigma_\epsilon}{\sqrt{1-\phi^2}} Z_0 \\ \sigma_\epsilon Z_1 \\ \sigma_\epsilon Z_2 \\ \vdots \\ \sigma_\epsilon Z_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ -\phi & 1 & 0 & \cdots \\ 0 & -\phi & 1 & \cdots \\ & \vdots & & \\ \cdots & 0 & -\phi & 1 \end{bmatrix} \left(\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_t \end{bmatrix} - \frac{c}{1-\phi} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right),$$

the Gaussian vector $(X_0, X_1, \dots, X_t)$ has density

$$p(x_0, x_1, \dots, x_t) = \left(2\pi \frac{\sigma_\epsilon^2}{1-\phi^2}\right)^{-\frac{1}{2}} \exp\left(-\frac{1-\phi^2}{2\sigma_\epsilon^2}\left(x_o - \frac{c}{1-\phi}\right)^2\right) \times \\ (2\pi\sigma_\epsilon^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}\sum_{s=1}^{t}(-\phi x_{s-1} + x_s - c)^2\right)$$

Note the product form

$$p(x_0, x_1, \ldots, x_t) = \left(2\pi \frac{\sigma_\epsilon^2}{1-\phi^2}\right)^{-\frac{1}{2}} \exp\left(-\frac{1-\phi^2}{2\sigma_\epsilon^2}\left(x_o - \frac{c}{1-\phi}\right)^2\right) \times$$
$$\prod_{s=1}^{t} (2\pi\sigma_\epsilon^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}\left(-\phi x_{s-1} + x_s - c\right)^2\right)$$

which, in turn, implies the Markov property

$$p_{X_t|X^{t-1}}(x_t|x^{t-1}) = \frac{p_{X_0,X_1,\ldots,X_t}(x_0, x_1, \ldots, x_t)}{p_{X_0,X_1,\ldots,X_{t-1}}(x_0, x_1, \ldots, x_{t-1})} =$$
$$(2\pi\sigma_\epsilon^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma_\epsilon^2}\left(-\phi x_{t-1} + x_t - c\right)^2\right) = p_{X_t|X_{t-1}}(x_t|x_{t-1})$$

let us compute the K matrix, that is the matrix of the quadratic form in the density above. If $y_s = x_s - c/(1-\phi)$ are the centered variables, the quadratic form is

$$\frac{1-\phi^2}{\sigma_\epsilon^2} y_0^2 + \frac{1}{\sigma_\epsilon^2}\sum_{s=1}^{t}\left(-\phi y_{s-1} + y_s\right)^2 =$$
$$\frac{1-\phi^2}{\sigma_\epsilon^2} y_0^2 + \frac{1}{\sigma_\epsilon^2}\sum_{s=1}^{t}\left(\phi^2 y_{s-1}^2 - 2\phi y_{s-1} y_s + y_s^2\right) =$$
$$\frac{1-\phi^2}{\sigma_\epsilon^2} y_0^2 + \frac{1}{\sigma_\epsilon^2}\left(\sum_{s=0}^{t-1}\phi^2 y_s^2 - \sum_{s=1}^{t} 2\phi y_{s-1} y_s + \sum_{s=1}^{t} y_s^2\right) =$$
$$\frac{1}{\sigma_\epsilon^2}\left(y_0^2 + \sum_{s=1}^{t-1}(1+\phi^2) y_s^2 + y_t^2 - 2\sum_{s=1}^{t}\phi y_{s-1} y_s\right),$$

hence

$$K(t) = \frac{1}{\sigma_\epsilon^2}\begin{bmatrix} 1 & -\phi & 0 & \ldots & 0 \\ -\phi & 1+\phi^2 & -\phi & \ldots & 0 \\ & \vdots & & & \\ 0 & \ldots & 0 & -\phi & 1 \end{bmatrix}.$$

Note the tri-diagonal form.

Let us compute the covariances $\Sigma(t) = K(t)^{-1}$. First, for each $t = 1, 2, \ldots$

$$\operatorname{Cov}\left(X_t, X_{t-1}\right) = \phi \operatorname{Cov}\left(X_{t-1}, X_{t-1}\right) + \sigma_\epsilon \operatorname{Cov}\left(Z_t, X_{t-1}\right) = \phi \frac{\sigma_\epsilon^2}{1-\phi^2},$$

which is constant for each t and, because of that, is called *autocovariance* at lag 1 $R(1)$. The same holds for each $R(k) = \operatorname{Cov}\left(X_t, X_{t-k}\right)$. By induction, for $k = 2, 3, \ldots$

$$\operatorname{Cov}\left(X_t, X_{t-k}\right) = \phi \operatorname{Cov}\left(, X_{t-1}\right) X_{t-k} + \sigma_\epsilon \operatorname{Cov}\left(Z_t, X_{t-k}\right) = \phi R(k-1)$$

so that $R(k) = \phi^k \frac{\sigma_\epsilon^2}{1-\phi^2}$, $k = 0, 1, \ldots$

1.2. **Variance.** The variance matrix of the vector $(X_0, X_1, \ldots, X_t)$ is

$$\begin{bmatrix} R(0) & R(1) & \cdots & R(t) \\ R(1) & R(0) & \cdots & R(t-1) \\ \vdots & & & \\ R(t) & R(t-1) & \cdots & R(0) \end{bmatrix} = \frac{\sigma_\epsilon^2}{1-\phi^2}\begin{bmatrix} \phi^0 & \phi^1 & \cdots & \phi^t \\ \phi^1 & \phi^0 & \cdots & \phi^{t-1} \\ \vdots & & & \\ \phi^t & \phi^{t-1} & \cdots & \phi^0 \end{bmatrix}$$

For example, we can check

$$\Sigma(2)K(2) =$$

$$\frac{\sigma_\epsilon^2}{1-\phi^2}\begin{bmatrix} 1 & \phi^1 & \phi^2 & \\ \phi^1 & \phi^0 & \phi^1 & \\ \phi^2 & \phi^1 & \cdots & 1 \end{bmatrix} \frac{1}{\sigma_\epsilon^2}\begin{bmatrix} 1 & -\phi & 0 \\ -\phi & 1+\phi^2 & -\phi \\ 0 & -\phi & 1 \end{bmatrix} = \frac{1}{1-\phi^2}\begin{bmatrix} 1-\phi^2 & 0 & 0 \\ 0 & 1-\phi^2 & 0 \\ 0 & 0 & 1-\phi^2 \end{bmatrix}$$

1.3. **Markov transition.** Let us compute the transitions of the Markov chain. The joint variance of (X_{t-1}, X_t) is $\frac{\sigma_\epsilon^2}{1-\phi^2}\begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$, the projection is $L = \frac{\sigma_\epsilon^2}{1-\phi^2}\phi\frac{1-\phi^2}{\sigma_\epsilon^2} = \phi$, the conditioned variance is $\frac{\sigma_\epsilon^2}{1-\phi^2}(1-\phi^2) = \sigma_\epsilon^2$, the conditional distribution is

$$X_t|(X_{t-1} = x_{t-1}) \sim \mathrm{N}_1\left(c + \phi x_{t-1}, \sigma_\epsilon^2\right) .$$

Let us redo the computation directly. For any $f \colon \mathbb{R} \to \mathbb{R}$

$$\begin{aligned}
\mathrm{E}\left(f(X_t)\middle|X^{t-1}\right) &= \mathrm{E}\left(f(c + \phi X_{-1} + Z_t)\middle|Z^{t-1}\right) \\
&= \int f(c + \phi X_{t-1} + z)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}\ dz \\
&= \int f(y)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(y-(c+\phi x))}\ dy\bigg|_{x=X_{t-1}} .
\end{aligned}$$

1.4. **Martingales.** Define $\mathcal{F}_t = \sigma(X^t) = \sigma(Z^t)$. $(Y_t)_{t\geqslant 0}$ is an L^2-martingale if $Y_t = f(t, X_0, \ldots, X_t)$ and $\mathrm{E}\left(Y_t|\mathcal{F}_{t-1}\right) = Y_{t-1}$. We have $X_t|X^{t-1} \sim X_t|X_{t-1}$, hence

$$\begin{aligned}
\mathrm{E}\left(f(t, X_0, \ldots, X_t)\middle|X^{t-1}\right) &= \mathrm{E}\left(f(t, x_0, \ldots, x_{t-1}, X_t)|X_{t-1}\right)|_{x^{t-1}=X^{t-1}} \\
&= \int f(t, x^{t-1}, x_t)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(x_t-(c+\phi x_{t-1}))^2}\ dx_t\bigg|_{x^{t-1}=X^{t-1}} \\
&= f(t-1, x^{t-1})\big|_{x^{t-1}=X^{t-1}}
\end{aligned}$$

2. Ornstein-Uhlembeck process

From the set of slides Wiener.pdf

Orstein-Uhlembeck process

Definition

Given a Wiener process W and constant $\theta, \sigma > 0$, $\mu \in \mathbb{R}$, the Ornstein-Uhlembeck process is $(X_t^x)_{t \geq 0}$ unique strong solution of

$$dX_t^x = \theta(\mu - X_t^x)dt + \sigma dW_t, \quad X_0^x = x$$

1. We focus on the standard case $dX_t^x = -X_t dt + \sqrt{2}W_t$, $X_0^x = 0$, precisely
$$X_t^x = x - \int_0^t X_u \, du + \sqrt{2}W_t, \quad t \geq 0$$
2. The unique strong solution is
$$\boxed{X_t^x = \mathrm{e}^{-t}x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)} \, dW_s}$$
where the stochastic integral is a Wiener integral.

Proof

1. $\mathrm{e}^t X_t^x + \mathrm{e}^t \int_0^t X_u \, du = \mathrm{e}^t x + \sqrt{2}\mathrm{e}^t W_t$
2. $\frac{d}{dt}\left(\mathrm{e}^t \int_0^t X_u \, du\right) = \mathrm{e}^t x + \sqrt{2}\mathrm{e}^t W_t$
3. $\mathrm{e}^t \int_0^t X_u \, du = (\mathrm{e}^t - 1)x + \sqrt{2}\int_0^t \mathrm{e}^s W_s \, ds$
4. $\int_0^t X_u \, du = (1 - \mathrm{e}^{-t})x + \sqrt{2}\mathrm{e}^{-t}\int_0^t \mathrm{e}^s W_s \, ds$
5.
$$\begin{aligned} X_t &= \mathrm{e}^{-t}x - \sqrt{2}\mathrm{e}^{-t}\int_0^t \mathrm{e}^s W_s \, ds + \sqrt{2}\mathrm{e}^{-t}\mathrm{e}^t W_t \\ &= \mathrm{e}^{-t}x + \sqrt{2}\mathrm{e}^{-t}\left(\mathrm{e}^t W_t - \int_0^t \mathrm{e}^s W_s \, ds\right) \\ &= \mathrm{e}^{-t}x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)} \, dW_s \end{aligned}$$

Distribution of the OU

- Each X_t^x is Gaussian with mean $\mathbb{E}(X_t^x) = \mathrm{e}^{-t}x$ and variance

$$\mathrm{Var}(X_t^x) = 2\,\mathbb{E}\left(\left(\int_0^t \mathrm{e}^{-(t-s)}\, dW_s\right)^2\right) =$$
$$\int_0^t 2\mathrm{e}^{-2(t-s)}\, ds = \mathrm{e}^{-2(t-s)}\Big|_{s=0}^{s=t} = 1 - \mathrm{e}^{-2t}$$

- For $t > s$ the covariance is

$$\begin{aligned}\mathrm{Cov}(X_t^x, X_s^x) &= 2\,\mathbb{E}\left(\left(\int_0^t \mathrm{e}^{-(t-u)}\, dW_u\right)\left(\int_0^s \mathrm{e}^{-(s-u)}\, dW_u\right)\right)\\ &= 2\int_0^s \mathrm{e}^{-(t-u)}\mathrm{e}^{-(s-u)}\, du\\ &= \mathrm{e}^{-s-t}\ \mathrm{e}^{2u}\Big|_{u=0}^{u=s}\\ &= \mathrm{e}^{-(t-s)} - \mathrm{e}^{-(t+s)} = \mathrm{e}^{-2(t-s)}(1-\mathrm{e}^{-2s})\end{aligned}$$

- OU $(X_t^x)_{t\geq 0}$ is a continuous adapted Gaussian process. Note that $\sigma(X_s^x : s \leq t) = \sigma(W_s^x : s \leq t)$.

OU is Markov I

- If $s < t$

$$\begin{bmatrix} X_s^x \\ X_t^x \end{bmatrix} \sim \mathrm{N}_2\left(\begin{bmatrix} \mathrm{e}^{-s} \\ \mathrm{e}^{-t} \end{bmatrix} x, \begin{bmatrix} 1-\mathrm{e}^{-2s} & \mathrm{e}^{-(t-s)}\mathrm{e}^{-2s} \\ \mathrm{e}^{-(t-s)}\mathrm{e}^{-2s} & 1-\mathrm{e}^{-2t} \end{bmatrix}\right)$$

hence

$$X_t^x | (X_s^x = y) \sim \mathrm{N}_1\left(\mathrm{e}^{-(t-s)y}, 1 - \mathrm{e}^{-2(t-s)}\right)$$

so that

$$\mathbb{E}\left(f(X_t^x)|(X_S^x = y)\right) = \int f(\mathrm{e}^{-(t-s)}y + \sqrt{1-\mathrm{e}^{-2(t-s)}}z)\gamma(dz)$$

- We have $X_t^x \sim \mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}Z$, hence we define the family of operators P_t, $t \geq 0$,

$$(P_t f)(x) = \mathbb{E}\left(f(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}Z)\right) =$$
$$\int f(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}z)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}\, dz$$

OU is Markov II

- Markov property:

$$
\begin{aligned}
&\mathbb{E}\left(f(X_t^x)|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(f(\mathrm{e}^{-t}x + \sqrt{2}\int_0^t \mathrm{e}^{-(t-s)}\, dW_s)\Big|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(f(\mathrm{e}^{-t}x + \sqrt{2}\int_0^s \mathrm{e}^{-(t-s)}\, dW_s + \sqrt{2}\int_s^t \mathrm{e}^{-(t-s)}\, dW_s)\Big|\mathcal{F}_s\right) \\
&= \int f(\mathrm{e}^{-t}x + \sqrt{2}\int_0^s \mathrm{e}^{-(t-s)}\, dW_s + \sqrt{1-\mathrm{e}^{-2(t-s)}z})\ \gamma(dz) \\
&= \int f(\mathrm{e}^{-(t-s)}X_s^x + \sqrt{1-\mathrm{e}^{-2(t-s)}z})\ \gamma(dz) \\
&= (P_{t-s}f)(X_s^x)
\end{aligned}
$$

- $(P_t)_{t\geq 0}$ is a semigroup, $P_s \circ P_{t-s} = P_t$ if $s \leq t$

References

1. Steffen L. Lauritzen, *Graphical models*, The Clarendon Press Oxford University Press, New York, 1996, Oxford Science Publications. MR 98g:62001
2. Ivan Nourdin and Giovanni Peccati, *Normal approximations with Malliavin calculus*, Cambridge Tracts in Mathematics, vol. 192, Cambridge University Press, Cambridge, 2012, From Stein's method to universality. MR 2962301

Home work

Read all the texts below, then choose and solve 2 exercises. The paper is due Fri May 29.

Exercise 1. The AR(1) process can be written see Autoregressive model in *Wikipedia*.

$$X_{t+1} = X_t + \theta(\mu - X_t) + Z_{t+1}, \quad |\theta| < 1$$

Review all properties in this notation.

Exercise 2. Compute the compensator and the quadratic variation of the AR(1) process.

Exercise 3. Prove that $L^2[0,1] \ni f \mapsto \int_0^1 f(s)\, dW_s$ is an isonormal Gaussian process, [NP] Section 2.1.

Exercise 4. Let W be a Wiener process.

(1) Find the variance $\Gamma(\boldsymbol{t})$ and the concentration $K(\boldsymbol{t})$ of finite the dimensional vectors $(W_{t_1}, \dots, W_{t_n})$, $0 < t_1 < \cdots < t_n$, $\boldsymbol{t} = (t_1, \cdots, t_n)$.

(2) Show that W is *Markov* with *kernel* $k(x,y) = \frac{1}{2\pi\sqrt{t-s}} \exp\left(-\frac{1}{2}\frac{(y-x)^2}{t-s}\right)$.

(3) Show that the processes W, $(W_t^2 - t)_{t>0}$, and $\left(\exp\left(aW_t - \frac{a^2}{2}t\right)\right)_{t\geqslant 0}$, $a \in \mathbb{R}$, are *martingales*.

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